

ON HOMOMORPHISMS BETWEEN CREMONA GROUPS

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ABSTRACT. We look at algebraic embeddings of the Cremona group in n variables $\mathrm{Cr}_n(\mathbb{C})$ to the groups of birational transformations $\mathrm{Bir}(M)$ of an algebraic variety M . First we study geometrical properties of an example of an embedding of $\mathrm{Cr}_2(\mathbb{C})$ into $\mathrm{Cr}_5(\mathbb{C})$ that is due to Gizatullin. In a second part, we give a full classification of all algebraic embeddings of $\mathrm{Cr}_2(\mathbb{C})$ into $\mathrm{Bir}(M)$, where $\dim(M) = 3$ and generalize this result partially to algebraic embeddings of $\mathrm{Cr}_n(\mathbb{C})$ into $\mathrm{Bir}(M)$, where $\dim(M) = n + 1$, for arbitrary n . In particular, this yields a classification of all algebraic $\mathrm{PGL}_{n+1}(\mathbb{C})$ -actions on smooth projective varieties of dimension $n + 1$ that can be extended to rational actions of $\mathrm{Cr}_n(\mathbb{C})$.

CONTENTS

1. Introduction and statement of the results	1
2. Algebraic homomorphisms	6
3. An example by Gizatullin	9
4. $\mathrm{PGL}_{n+1}(\mathbb{C})$ -actions in codimension 1	20
5. Extension to Cr_n	23
Appendix	29
References	31

1. INTRODUCTION AND STATEMENT OF THE RESULTS

1.1. **Cremona groups.** Let M be a complex variety and $\mathrm{Bir}(M)$ the group of birational transformations of M . Denote by $\mathbb{P}^n = \mathbb{P}_{\mathbb{C}}^n$ the complex projective space of dimension n . The group

$$\mathrm{Cr}_n := \mathrm{Bir}(\mathbb{P}^n)$$

is called the *Cremona group*. In this paper we are interested in group homomorphisms from Cr_n to $\mathrm{Bir}(M)$. In particular, we will study an embedding of Cr_2 into Cr_5 that was described by Gizatullin [Giz99] and consider the case, where $\dim(M) = n + 1$.

A birational transformation $A: M \dashrightarrow N$ between varieties M and N induces an isomorphism $\mathrm{Bir}(M) \rightarrow \mathrm{Bir}(N)$ by conjugating elements of $\mathrm{Bir}(M)$ with A . Two homomorphisms $\Phi: \mathrm{Bir}(M) \rightarrow \mathrm{Bir}(N_1)$ and $\Psi: \mathrm{Bir}(M) \rightarrow \mathrm{Bir}(N_2)$ are called

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conjugate if there exists a birational transformation $A: N_1 \dashrightarrow N_2$ such that $\Psi(g) = A \circ \Phi(g) \circ A^{-1}$ for all $g \in \text{Bir}(M)$.

Example 1.1. Assume that a variety M is birationally equivalent to $\mathbb{P}^n \times N$ for some variety N . The standard action on the first factor yields an injective homomorphism of Cr_n into $\text{Bir}(\mathbb{P}^n \times N)$ and therefore also into $\text{Bir}(M)$. We call embeddings of this type *standard embeddings*. In particular, we obtain in that way for all nonnegative integers m an injective homomorphism $\text{Cr}_n \rightarrow \text{Bir}(\mathbb{P}^n \times \mathbb{P}^m)$.

Example 1.2. A variety M is called *stably rational* if there exists a n such that $M \times \mathbb{P}^n$ is rational. There exist varieties of dimension larger than or equal to 3 that are stably rational but not rational (see [BCTSSD85]). We will see that two standard embeddings $f_1: \text{Cr}_n \rightarrow \text{Bir}(\mathbb{P}^n \times N)$ and $f_2: \text{Cr}_n \rightarrow \text{Bir}(\mathbb{P}^n \times M)$ are conjugate if and only if N and M are birationally equivalent (Lemma 3.3). So every class of birationally equivalent stably rational varieties of dimension k defines a different conjugacy class of injective homomorphisms $\text{Cr}_n \rightarrow \text{Bir}(\mathbb{P}^m)$ for $m = n + k$.

1.2. Notation and subgroups of Cr_n . If we fix homogeneous coordinates $[x_0 : \dots : x_n]$ of \mathbb{P}^n , every element $f \in \text{Cr}_n$ can be described by homogeneous polynomials of the same degree $f_0, \dots, f_n \in \mathbb{C}[x_0, \dots, x_n]$ without non-constant common factor, such that

$$f([x_0 : \dots : x_n]) = [f_0 : \dots : f_n].$$

The *degree* of f is the degree of the f_i .

With respect to affine coordinates $[1 : X_1 : \dots : X_n] = (X_1, \dots, X_n)$, we have

$$f(X_1, \dots, X_n) = (F_1, \dots, F_n),$$

where $F_i(X_1, \dots, X_n) = f_i(1, X_1, \dots, X_n)/f_0(1, X_1, \dots, X_n) \in \mathbb{C}(X_1, \dots, X_n)$.

An important subgroup of Cr_n is the automorphism group

$$\text{Aut}(\mathbb{P}^n) \simeq \text{PGL}_{n+1}(\mathbb{C}).$$

The n -dimensional subgroup of $\text{Aut}(\mathbb{P}^n)$ consisting of diagonal automorphisms will be denoted by D_n .

Let $A = (a_{ij}) \in M_n(\mathbb{Z})$ be a matrix of integers. The matrix A determines a rational self map of the affine space

$$f_A = (x_1^{a_{11}} x_2^{a_{12}} \dots x_n^{a_{1n}}, x_1^{a_{21}} x_2^{a_{22}} \dots x_n^{a_{2n}}, \dots, x_1^{a_{n1}} x_2^{a_{n2}} \dots x_n^{a_{nn}}).$$

We have $f_A \circ f_B = f_{AB}$ for $A, B \in M_n(\mathbb{Z})$. One observes that f_A is a birational transformation if and only if $A \in \text{GL}_n(\mathbb{Z})$. This yields an injective homomorphism $\text{GL}_n(\mathbb{Z}) \rightarrow \text{Cr}_n$ whose image we call the *Weyl group* and denote it by W_n . This terminology is justified by the fact that the normalizer of D_n in Cr_n is the semidirect product $\text{Norm}_{\text{Cr}_n}(D_n) = D_n \rtimes W_n$. Note that $D_n \rtimes W_n$ is the automorphism group of $(\mathbb{C}^*)^n$. Sometimes, W_n is also called the *group of monomial transformations*.

The Cremona group Cr_n contains $\text{Aut}(\mathbb{A}^n)$, the group of polynomial automorphisms of the affine space \mathbb{A}^n . We always consider the embedding of $\text{Aut}(\mathbb{A}^n)$ into Cr_n by considering the affine coordinates given by $x_0 \neq 0$.

1.3. Previous results. The well known theorem of Noether and Castelnuovo (see for example [AC02]) states that over an algebraically closed field k the Cremona

group in two variables is generated by $\mathrm{PGL}_3(k)$ and the standard quadratic involution

$$\sigma := [x_1x_2 : x_0x_2 : x_0x_1] \in W_2.$$

Results of Hudson and Pan ([Hud27], [Pan99]) show that for $n \geq 3$ the Cremona group Cr_n is not generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and W_n . Let

$$H_n := \langle \mathrm{PGL}_{n+1}(\mathbb{C}), W_n \rangle.$$

Blanc and Hedén studied the subgroup G_n of Cr_n generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and the element $\sigma_n := [x_0^{-1} : \dots : x_n^{-1}]$ ([BH14]). In particular, they show that G_n is strictly contained in H_n if and only if n is odd. Further results about the group structure of G_n can be found in [Dés14].

Let $\gamma: \mathbb{C} \rightarrow \mathbb{C}$ be an automorphism of fields. By acting on the coordinates, γ induces a bijective map $\Gamma: \mathbb{P}^n \rightarrow \mathbb{P}^n$. Conjugation with Γ yields a group automorphism of Cr_n that preserves degrees. Observe that we obtain the image of $g \in \mathrm{Cr}_n$ by letting γ operate on the coefficients of g . By abuse of notation we denote this automorphism by γ as well. In [Dés06b] Déserti showed that all automorphisms of Cr_2 are inner up to such field automorphisms. A generalization of this result is the following theorem by Cantat:

Theorem 1.3 ([Can14]). *Let M be a smooth projective variety of dimension n and $r \in \mathbb{Z}^+$. Let*

$$\rho: \mathrm{PGL}_{r+1}(\mathbb{C}) \rightarrow \mathrm{Bir}(M)$$

be a non-trivial group homomorphism. Then $n \geq r$ and if $n = r$ then M is rational and there exists an automorphism of fields $\gamma: \mathbb{C} \rightarrow \mathbb{C}$ such that $\rho \circ \gamma$ is conjugate to the standard embedding of $\mathrm{PGL}_{n+1}(\mathbb{C})$ into Cr_n .

In the Appendix we will prove two corollaries of Theorem 1.3 that show some implications of this result to group endomorphisms of Cr_n .

1.4. Algebraic homomorphisms. We call a group homomorphism $\Psi: \mathrm{Cr}_n \rightarrow \mathrm{Bir}(M)$ *algebraic* if its restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is an algebraic morphism. The algebraic structure of $\mathrm{Bir}(M)$ and some properties of algebraic homomorphisms will be discussed in Section 2. Recall that an element $f \in \mathrm{Cr}_n$ is called *algebraic*, if the sequence $\{\deg(f^n)\}_{n \in \mathbb{Z}^+}$ is bounded.

1.5. Reducibility.

Definition. Let M be a variety $\varphi_M: \mathrm{Cr}_n \rightarrow \mathrm{Bir}(M)$ a non-trivial algebraic group homomorphism. We say that φ_M is *reducible* if there exists a variety N such that $0 < \dim(N) < \dim(M)$ and an algebraic homomorphism $\varphi_N: \mathrm{Cr}_n \rightarrow \mathrm{Bir}(N)$ together with a dominant rational map $\pi: M \dashrightarrow N$ that is Cr_2 -equivariant with respect to the rational actions induced by φ_M and φ_N respectively.

Remark 1. In [Zha10], Zhang uses the terminology *primitive action* for irreducible actions in the sense of Definition 1.5; in [Can03], Cantat says that an action *admits a non-trivial factor* if it is reducible.

Note that if we look at the induced action of Cr_n on the function field $\mathbb{C}(M)$ of M , reducibility is equivalent to the existence of a Cr_n -invariant function field $\mathbb{C}(N) \subset \mathbb{C}(M)$.

1.6. An example by Gizatullin. In [Giz99], Gizatullin looks at the following question: Let $\psi: \mathrm{PGL}_3(\mathbb{C}) \rightarrow \mathrm{PGL}_{n+1}(\mathbb{C})$ be a linear representation. Does ψ extend to a homomorphism $\Psi: \mathrm{Cr}_2 \rightarrow \mathrm{Cr}_n$? He shows that the linear representations given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on conics, cubics and quartics can be extended to homomorphisms from Cr_2 to Cr_5 , Cr_9 and Cr_{14} , respectively.

In Section 3 we study in detail some geometrical properties of the homomorphism

$$\Phi: \mathrm{Cr}_2 \rightarrow \mathrm{Cr}_5$$

that was described by Gizatullin; by construction, the restriction of Φ to PGL_3 yields the linear representation $\varphi: \mathrm{PGL}_3(\mathbb{C}) \rightarrow \mathrm{PGL}_6(\mathbb{C})$ given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on plane conics. Among other things, we prove the following:

Theorem 1.4. *Let $\Phi: \mathrm{Cr}_2 \rightarrow \mathrm{Cr}_5$ be the Gizatullin homomorphism. Then the following is true:*

- (1) *The group homomorphism Φ is injective and irreducible.*
- (2) *The rational action of Cr_2 on \mathbb{P}^5 that is induced by Φ preserves the Veronese surface V and its secant variety $S \subset \mathbb{P}^5$ and induces rational actions of Cr_2 on V and S .*
- (3) *The Veronese embedding $v: \mathbb{P}^2 \rightarrow \mathbb{P}^5$ is Cr_2 -equivariant with respect to the standard rational action on \mathbb{P}^2 .*
- (4) *The surjective secant morphism $s: \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow S \subset \mathbb{P}^5$ (see Section 3.4) is Cr_2 -equivariant with respect to the diagonal action of Cr_2 on $\mathbb{P}^2 \times \mathbb{P}^2$.*
- (5) *The rational action of Cr_2 on \mathbb{P}^5 preserves a volume form on \mathbb{P}^5 with poles of order three along the secant variety S .*
- (6) *The group homomorphism Φ sends the group of polynomial automorphisms $\mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2$ to $\mathrm{Aut}(\mathbb{A}^5)$.*

Note that the injectivity of Φ follows from (3); in Section 3.8 irreducibility is proved. Part (2) - (4) of Theorem 1.4 will be proved in Section 3.4, part (5) in Section 3.6 and part (6) in Section 3.7.

The representation φ^\vee of PGL_3 into PGL_6 given by $\psi \circ \alpha$, where α is the algebraic homomorphism $g \mapsto {}^t g^{-1}$, is conjugate in Cr_5 to the representation φ . This conjugation yields the embedding $\Phi^\vee: \mathrm{Cr}_2 \rightarrow \mathrm{Cr}_5$, whose image preserves the secant variety S as well and induces a rational action on it. As the secant variety S is rational, Φ and Φ^\vee induce two non-standard embeddings of Cr_2 into Cr_4 , which we denote by Ψ_1 and Ψ_2 respectively. In Section 3.5 we prove the following:

Proposition 1.5. *The two embeddings $\Psi_1, \Psi_2: \mathrm{Cr}_2 \rightarrow \mathrm{Cr}_4$ are not conjugate in Cr_4 ; moreover they are irreducible and therefore not conjugate to the standard embedding.*

Proposition 1.5 shows in particular that there exist at least three different embeddings of Cr_2 into Cr_4 .

Since Φ is algebraic, the images of algebraic elements under Φ are algebraic again (see Proposition 2.3). Calculation of the degrees of some examples suggests that Φ might even preserve the degrees of all elements in Cr_2 . However, we were only able to prove the following (Section 3.7):

Theorem 1.6. *Let $\Phi: \mathrm{Cr}_2 \rightarrow \mathrm{Cr}_5$ be the Gizatullin-embedding. Then*

- (1) *for all elements $f \in \mathrm{Cr}_2$ we have $\deg(f) \leq \deg(\Phi(f))$,*
- (2) *for all $g \in \mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2$ we have $\deg(g) = \deg(\Phi(g))$.*

The image of the Weyl group W_2 under Φ is not contained in the Weyl group W_5 . *More generally, it can be shown that there exists no algebraic homomorphism from Cr_2 to Cr_5 that preserves automorphisms, diagonal automorphisms and the Weyl group* (see [Ureon]).

1.7. Algebraic embeddings in codimension 1. In Section 4 and Section 5 we look at algebraic homomorphisms $\text{Cr}_n \rightarrow \text{Bir}(M)$ in the case where M is a smooth projective variety of dimension $n + 1$ for $n \geq 2$.

Example 1.7. For all curves C of genus ≥ 1 , the variety $\mathbb{P}^n \times C$ is not rational and there exists the standard embedding $\Psi_C: \text{Cr}_n \rightarrow \text{Bir}(\mathbb{P}^n \times C)$.

Example 1.8. Cr_n acts rationally on the total space of the canonical bundle of \mathbb{P}^n

$$K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)) \simeq \bigwedge^n (T\mathbb{P}^n)^\vee$$

by $f(p, \omega) = (f(p), \omega \circ (df_p)^{-1})$, where $p \in \mathbb{P}^n$ and $\omega \in \bigwedge^n (T_p \mathbb{P}^2)^\vee$. This action extends to the projective completion

$$F_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-(n+1))).$$

More generally, we obtain an action of Cr_n on the total space of the bundle $K_{\mathbb{P}^n}^{\otimes l} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)l)$ and on its projective completion

$$F_l := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-l(n+1)))$$

for all $l \in \mathbb{Z}_{\geq 0}$. This yields a countable family of injective homomorphisms

$$\Psi_l: \text{Cr}_n \rightarrow \text{Bir}(F_l).$$

We can choose affine coordinates $(x_1, \dots, x_n, x_{n+1})$ of F_l such that Ψ_l is given by

$$\Psi_l(f)(x_1, \dots, x_n, x_{n+1}) = (f(x_1, \dots, x_n), J(f(x_1, \dots, x_n))^{-l} x_{n+1}).$$

Here, $J(f(x_1, \dots, x_n))$ denotes the determinant of the Jacobian of f at the point (x_1, \dots, x_n) . Observe that Ψ_0 is conjugate to the standard embedding.

Example 1.9. Let $\mathbb{P}(T\mathbb{P}^2)$ be the total space of the fiberwise projectivisation of the tangent bundle over \mathbb{P}^2 . Then $\mathbb{P}(T\mathbb{P}^2)$ is rational and there is an injective group homomorphism

$$\Psi_B: \text{Cr}_2 \rightarrow \text{Bir}(\mathbb{P}(T\mathbb{P}^2))$$

defined by $\Psi_B(f)(p, v) := (f(p), \mathbb{P}(df_p)(v))$. Here, $\mathbb{P}(df_p): \mathbb{P}T_p \rightarrow \mathbb{P}T_{f(p)}$ defines the projectivisation of the differential df_p of f at the point $p \in \mathbb{P}^2$.

Example 1.10. The Grassmannian of lines in the projective 3-space $\mathbb{G}(1, 3)$ is a rational variety of dimension 4 with a transitive algebraic $\text{PGL}_4(\mathbb{C})$ -action. This action induces an algebraic embedding of $\text{PGL}_4(\mathbb{C})$ into Cr_4 . In Proposition 5.2 we will show that the image of this embedding does not lie in any subgroup isomorphic to Cr_3 . *So no group action of $\text{PGL}_4(\mathbb{C})$ on $\mathbb{G}(1, 3)$ by automorphisms can be extended to a rational action of Cr_3 .*

The classification of PGL_{n+1} -actions on smooth projective varieties of dimension $n + 1$ is well known to the experts; in Section 4 we study their conjugacy classes. In fact, we will see that Examples 1.7 to 1.10 describe up to birational conjugation and up to algebraic homomorphisms of PGL_{n+1} all possible PGL_{n+1} -actions on smooth projective varieties of dimension $n + 1$ and that these actions are not birationally

conjugate to each other. This yields a classification of algebraic homomorphisms of PGL_{n+1} to $\mathrm{Bir}(M)$. We will study in Section 5 how these actions extend to rational actions of Cr_n on M .

Theorem 1.11. *Let $n \geq 2$ and let M be a complex projective variety of dimension $n + 1$ and let $\varphi: \mathrm{PGL}_{n+1}(\mathbb{C}) \rightarrow \mathrm{Bir}(M)$ be a non-trivial algebraic homomorphism, then φ is conjugate to one of the embeddings described in Example 1.7 to 1.9. If φ is not conjugate to the action described in Example 1.10, then there exists up to conjugation a unique algebraic homomorphism α of $\mathrm{PGL}_{n+1}(\mathbb{C})$ such that $\varphi \circ \alpha$ extends to a homomorphism of Cr_n to $\mathrm{Bir}(M)$. Moreover, this extension is unique if restricted to the subgroup $H_n = \langle \mathrm{PGL}_{n+1}(\mathbb{C}), W_n \rangle \subset \mathrm{Cr}_n$.*

Theorem 1.11 classifies all group homomorphisms $\Psi: H_n \rightarrow \mathrm{Bir}(M)$ for projective varieties M of dimension $n + 1$ such that the restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a morphism. By the theorem of Noether and Castelnuovo, we obtain in particular a full classification of all algebraic homomorphisms from Cr_2 to $\mathrm{Bir}(M)$ for projective varieties M of dimension 3:

Corollary 1.12. *Let M be a projective variety of dimension 3 and $\Psi: \mathrm{Cr}_2 \rightarrow \mathrm{Bir}(M)$ a non-trivial algebraic group homomorphism. Then Ψ is conjugate to exactly one of the homomorphisms described in Example 1.7 to 1.9.*

The following observations are now immediate:

Corollary 1.13. *Let M be a projective variety of dimension 3 and $\Psi: \mathrm{Cr}_2 \rightarrow \mathrm{Bir}(M)$ a non-trivial algebraic homomorphism. Then*

- (1) Ψ is injective.
- (2) There exists a Cr_2 -equivariant rational map $f: M \dashrightarrow \mathbb{P}^2$ with respect to the rational action induced by Ψ and the standard action respectively. In particular, all algebraic homomorphisms from Cr_2 to $\mathrm{Bir}(M)$ are reducible.
- (3) There exists an integer $C_\Psi \in \mathbb{Z}$ such that

$$1/C_\Psi \deg(f) \leq \deg(\Psi(f)) \leq C_\Psi \deg(f).$$

Note that Part (3) of Corollary 1.13 resembles in some way Theorem 1.6. It seems to be an interesting question how the degree of the image of an element $f \in \mathrm{Cr}_2$ under an algebraic homomorphism is related to the degree of f .

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2. ALGEBRAIC HOMOMORPHISMS

In this section we recall some results on the algebraic structure of $\mathrm{Bir}(M)$ and of some of its subgroups and we discuss our notion of algebraic homomorphisms.

2.1. The Zariski topology. We can equip $\mathrm{Bir}(M)$ with the so-called Zariski topology. Let A be an algebraic variety and

$$f: A \times M \dashrightarrow A \times M$$

an A -birational map inducing an isomorphism between open subsets U and V of $A \times M$ such that the projections from U and from V to A are both surjective. For each $a \in A$ we obtain therefore an element of $\mathrm{Bir}(M)$ defined by $x \mapsto p_2(f(a, x))$,

where p_2 is the second projection. Such a map $A \rightarrow \text{Bir}(M)$ is called a *morphism* or *family of birational transformations parametrized by A* .

Definition. The Zariski topology on $\text{Bir}(M)$ is the finest topology such that all morphisms $f: A \rightarrow \text{Bir}(M)$ for all algebraic varieties A are continuous (with respect to the Zariski topology on A).

The map $\iota: \text{Bir}(M) \rightarrow \text{Bir}(M)$, $x \mapsto x^{-1}$ is continuous as well as the maps $x \mapsto g \circ x$ and $x \mapsto x \circ g$ for any $g \in \text{Bir}(M)$. This follows from the fact that the inverse of an A -birational map as above is again an A -birational map as is the right/left-composition with an element of $\text{Bir}(M)$. The Zariski topology was introduced in [Dem70] and [Ser08] and studied in [BF13].

2.2. Algebraic subgroups. An algebraic subgroup of $\text{Bir}(M)$ is the image of an algebraic group G by a morphism $G \rightarrow \text{Bir}(M)$ that is also an injective group homomorphism. It can be shown that algebraic groups are closed in the Zariski topology and of bounded degree in the case of $\text{Bir}(M) = \text{Cr}_n$. Conversely, closed subgroups of bounded degree in Cr_n are always algebraic subgroups with a unique algebraic group structure that is compatible with the Zariski topology (see [BF13]).

Let N be a smooth projective variety that is birationally equivalent to M . Let G be an algebraic group acting regularly and faithfully on N . This yields a morphism $G \rightarrow \text{Bir}(M)$, so G is an algebraic subgroup of $\text{Bir}(M)$. On the other hand, a theorem by Weil states that all algebraic subgroups of $\text{Bir}(M)$ have this form.

Theorem 2.1 ([Wei55], [Sum74], [Zai95]). *Let $G \subset \text{Bir}(M)$ be an algebraic subgroup. Then there exists a smooth projective variety N and a birational map $f: M \dashrightarrow N$ that conjugates G to a subgroup of $\text{Aut}(N)$ such that the induced action on N is algebraic.*

It can be shown (see for example, [BF13]) that the sets $(\text{Cr}_n)_{\leq d} \subset \text{Cr}_n$ consisting of all birational transformations of degree $\leq d$ are closed with respect to the Zariski topology. So the closure of a subgroup of bounded degree in Cr_n is an algebraic subgroup and can therefore be regularized. We obtain:

Corollary 2.2. *Let $G \subset \text{Cr}_n$ be a subgroup that is contained in some $(\text{Cr}_n)_{\leq d}$, then there exists a smooth projective variety N and a birational transformation $f: \mathbb{P}^n \dashrightarrow N$ such that $fGf^{-1} \subset \text{Aut}(N)$.*

The maximal algebraic subgroups of Cr_2 have been classified together with the rational surfaces on which they act as automorphisms ([Enr93], [Bla09]). In dimension 3, a classification for maximal connected algebraic subgroups exists: [Ume82b], [Ume85], [Ume82a].

2.3. Algebraic homomorphisms and continuous homomorphisms. We defined a group homomorphism from Cr_n to $\text{Bir}(M)$ to be algebraic if its restriction to $\text{PGL}_{n+1}(\mathbb{C})$ is a morphism. Note that this is a priori a weaker notion than being continuous with respect to the Zariski topology. It is not clear, whether algebraic homomorphisms are always continuous. However, for dimension 2 we have the following partial result, which will be proved in Section 2.5:

Proposition 2.3. *Let $\Phi: \text{Cr}_2 \rightarrow \text{Bir}(M)$ be a homomorphism of groups. The following are equivalent:*

- (1) Φ is algebraic.

- (2) *The restriction of Φ to any algebraic subgroup of Cr_2 is algebraic.*
- (3) *The restriction of Φ to one positive dimensional algebraic subgroup of Cr_2 is algebraic.*

2.4. One-parameter subgroups. A one-parameter subgroup is a connected algebraic group of dimension 1. It is well known (see for example [Hum75]) that all one-parameter subgroups are isomorphic to either \mathbb{C} or \mathbb{C}^* . The group \mathbb{C} is unipotent, the group \mathbb{C}^* semi-simple.

Proposition 2.4 shows that, up to conjugation by birational maps, there exists only one birational action of \mathbb{C} and only one of \mathbb{C}^* on \mathbb{P}^2 :

Proposition 2.4. *In Cr_2 all one-parameter subgroups isomorphic to \mathbb{C} are conjugate and all one-parameter subgroups isomorphic to \mathbb{C}^* are conjugate.*

The first part of Proposition 2.4 follows from results in [BD15] and [Bla06] (see also [Bru97]). The second part is a special case of Theorem 2.5. A detailed explanation of the proof can be found in [Ureon].

Theorem 2.5 ([BB66], [Pop13]). *In Cr_n all tori of dimension $\geq n-2$ are conjugate to a subtorus of D_n . Moreover, two subtori of D_n are conjugate in Cr_n to each other if and only if they are isomorphic.*

The following Lemma is a classical result (see for example [Sta13]):

Lemma 2.6. *Let G be a linear algebraic group and U_1, \dots, U_n be algebraic subgroups such that $U_1 U_2 \cdots U_n = G$. Let H be a linear algebraic group and $\varphi: G \rightarrow H$ a homomorphism of abstract groups such that $\varphi|_{U_i}$ is a homomorphism of algebraic groups for all i . Then φ is a homomorphism of algebraic groups.*

2.5. Algebraic and abstract group homomorphisms. Let G and H be algebraic groups that are isomorphic as abstract groups. The question whether G and H are also isomorphic as algebraic groups have been treated in detail in [BT73] (see also [Die71] and [Dés06a]). We will use the following result:

Proposition 2.7. *Let G be an algebraic group that is isomorphic to $\mathrm{PGL}_n(\mathbb{C})$ as an abstract group. Then G is isomorphic to $\mathrm{PGL}_n(\mathbb{C})$ as an algebraic group. Moreover, for every abstract isomorphism*

$$\rho: \mathrm{PGL}_n(\mathbb{C}) \rightarrow G$$

there exists an automorphism of fields $\tau: \mathbb{C} \rightarrow \mathbb{C}$ such that $\rho \circ \tau$ is an algebraic isomorphism.

Remark 2. It is well known that the automorphisms of $\mathrm{PGL}_n(\mathbb{C})$ as an algebraic group are compositions of inner automorphisms and the automorphism

$$\alpha: \mathrm{PGL}_n(\mathbb{C}) \rightarrow \mathrm{PGL}_n(\mathbb{C}), \quad g \mapsto {}^t g^{-1}.$$

Proof of Proposition 2.3. We first show how (1) implies (2). Let G be an algebraic subgroup of Cr_2 . We can assume that G is connected. There exist one parameter subgroups $U_1, \dots, U_k \subset G$ such that $U_1 \cdots U_k = G$ and there exists a constant C such that every element in G can be written as the product of at most C elements of $U_1 \cup U_2 \cup \dots \cup U_n$. Since, by Proposition 2.4, the group U_i is conjugate to a one parameter subgroup of $\mathrm{PGL}_3(\mathbb{C})$ for all i , we obtain that the restriction of φ to any of the U_i is an algebraic homomorphism of groups and that $\varphi(G) \subset \mathrm{Cr}_n$ is of bounded degree. Then $\overline{\varphi(G)} \subset \mathrm{Cr}_n$ is an algebraic group. We can now apply

Lemma 2.6 and conclude that the restriction of φ to G is a homomorphism of algebraic groups.

Statement (3) follows immediately from statement (2), so it only remains to prove that (3) implies (1). Let $\varphi: \text{Cr}_2 \rightarrow \text{Bir}(M)$ be a homomorphism of abstract groups and let $G \subset \text{Cr}_2$ be a positive dimensional algebraic subgroup such that the restriction of φ to G is a morphism. Since G is infinite, it contains a one parameter subgroup $U \subset G$.

Let $U_1, \dots, U_n \subset \text{PGL}_3(\mathbb{C})$ be unipotent one parameter subgroups such that $U_1 \cdots U_n = \text{PGL}_3(\mathbb{C})$ and C a constant such that every element in $\text{PGL}_3(\mathbb{C})$ can be written as the product of at most C elements of $U_1 \cup U_2 \cup \cdots \cup U_n$. If U is unipotent, all the subgroups U_i are conjugate to U . Hence the restriction of φ to U_i is a morphism for all i . The image $\varphi(\text{PGL}_3(\mathbb{C})) \subset \text{Cr}_n$ is of bounded degree, so $\overline{\varphi(\text{PGL}_3(\mathbb{C}))} \subset \text{Cr}_n$ is an algebraic group and with Lemma 2.6 it follows that the restriction of φ to $\text{PGL}_3(\mathbb{C})$ is a morphism.

Denote by $D_1 \subset \text{PGL}_3(\mathbb{C})$ the subgroup given by elements of the form $[cx_0 : x_1 : x_2]$, $c \in \mathbb{C}^*$ and by $T \subset \text{PGL}_3(\mathbb{C})$ the subgroup of all elements of the form $[x_0 : x_1 + cx_0 : x_2]$, $c \in \mathbb{C}$; we have $D_1 \simeq \mathbb{C}^*$ and $T \simeq \mathbb{C}$. If U is semi-simple, it is, again by Proposition 2.4, conjugate to D_1 , hence the restriction of φ to D_1 is a morphism well. Note that

$$T = \{[x_0 : x_1 + cx_0 : x_2] \mid c \in \mathbb{C}\} = \{dgd^{-1} \mid d \in D_1\} \cup \{\text{id}\}$$

where $g = [x_0 : x_1 + x_0 : x_2]$. We obtain that $\varphi(T)$ is of bounded degree and contained in the algebraic group $\overline{\varphi(T)} \subset \text{Cr}_n$. As $\varphi(T)$ consists of two $\varphi(D_1)$ -orbits, it is constructible and therefore closed. We obtain that the images of all unipotent subgroups of Cr_2 under φ are algebraic subgroups. The map $\varphi(U_1) \times \cdots \times \varphi(U_n) \rightarrow \text{Cr}_n$ is a morphism, so its image is a constructible set and therefore closed since it is a group. Hence $\varphi(\text{PGL}_3(\mathbb{C})) = \varphi(U_1) \cdots \varphi(U_n)$ is an algebraic subgroup. By Proposition 2.7 it is isomorphic as an algebraic group to $\text{PGL}_3(\mathbb{C})$ and there exists an automorphism of fields $\tau: \mathbb{C} \rightarrow \mathbb{C}$ such that $\varphi \circ \tau: \text{PGL}_3(\mathbb{C}) \rightarrow \text{PGL}_3(\mathbb{C})$ is an isomorphism of algebraic groups. But since the restriction of φ to T is already an algebraic homomorphism, it follows that τ is the identity. \square

Remark 3. Proposition 2.3 shows in particular that algebraic homomorphisms $\Psi: \text{Cr}_2 \rightarrow \text{Bir}(M)$ send algebraic elements to algebraic elements. This result follows also directly from the fact that a birational transformation $f \in \text{Cr}_2$ of degree d can be written as the product of at most $4d$ linear maps and $4d$ times the standard quadratic involution σ (see for example [AC02]); we therefore obtain that the sequence $\{\deg(\Phi(f)^n)\}$ is bounded if $\{\deg(f^n)\}$ is bounded.

3. AN EXAMPLE BY GIZATULLIN

3.1. Projective representations of the projective linear group. The results from representation theory of linear algebraic groups that we use in this section can be found, for example, in [FH91], [Pro07].

Proposition 3.1. *There is a bijection between homomorphisms of algebraic groups from $\text{SL}_n(\mathbb{C})$ to $\text{SL}_m(\mathbb{C})$ such that the image of the center is contained in the center and homomorphisms of algebraic groups from $\text{PGL}_n(\mathbb{C})$ to $\text{PGL}_m(\mathbb{C})$.*

From Proposition 3.1 and some elementary representation theory of $\mathrm{SL}_3(\mathbb{C})$ it follows that $n = 6$ is the smallest number such that there exist non-trivial and non-standard homomorphisms of algebraic groups from $\mathrm{PGL}_3(\mathbb{C})$ to $\mathrm{PGL}_n(\mathbb{C})$. In fact, up to automorphisms of $\mathrm{PGL}_3(\mathbb{C})$ there are exactly two non-trivial representations from $\mathrm{PGL}_3(\mathbb{C})$ to $\mathrm{PGL}_6(\mathbb{C})$.

The first one is reducible. Let $\psi': \mathrm{GL}_3 \rightarrow \mathrm{GL}_6$ be the linear representation given by the diagonal action on $\mathbb{C}^3 \times \mathbb{C}^3$; we denote by $\psi: \mathrm{PGL}_3(\mathbb{C}) \rightarrow \mathrm{PGL}_6(\mathbb{C})$ its projectivisation.

The second one is given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on the space of conics. The latter one can be parametrized by the space $\mathbb{P}M_3$ of symmetric 3×3 -matrices up to scalar multiple and is isomorphic to \mathbb{P}^5 . Let $g \in \mathrm{PGL}_3(\mathbb{C})$, we define $\varphi(g) \in \mathrm{PGL}_6(\mathbb{C})$ by $(a_{ij}) \mapsto g(a_{ij})({}^t g)$.

In this section we identify the space of conics with \mathbb{P}^5 in the following way:

$$(a_{ij}) \mapsto [a_{00} : a_{11} : a_{22} : a_{12} : a_{02} : a_{01}]$$

In other words, the conic C given by the zeroes of the equation

$$F = a_{00}X^2 + a_{11}Y^2 + a_{22}Z^2 + 2a_{12}YZ + 2a_{02}XZ + 2a_{01}XY$$

is identified with the point $[a_{00} : a_{11} : a_{22} : a_{12} : a_{02} : a_{01}] \in \mathbb{P}^5$.

Observe that with our definition, $\varphi(g)$ sends the conic C to the conic given by the zero set of the polynomial $F \circ ({}^t g)$.

Let

$$\alpha: \mathrm{PGL}_3(\mathbb{C}) \rightarrow \mathrm{PGL}_3(\mathbb{C})$$

be the algebraic automorphism $g \mapsto ({}^t g)^{-1}$. Then $\varphi(\alpha(g))$ maps the conic C to $g(C)$, which is the conic given by the zero set of the polynomial $F \circ g^{-1}$. Accordingly, $\varphi(\alpha(g)) \in \mathrm{PGL}_6(\mathbb{C})$ maps the matrix $(a_{ij}) \in M_3$ to $({}^t g)^{-1}(a_{ij})g^{-1}$.

The action of $\mathrm{PGL}_3(\mathbb{C})$ on \mathbb{P}^5 induced by φ has exactly three orbits that are characterized by the rank of the corresponding symmetric matrix in M_3 . Geometrically they correspond to the sets of smooth conics, pairs of distinct lines and double lines. The set of double lines is a surface isomorphic to \mathbb{P}^2 and called the *Veronese surface*; we denote it by V . The set of singular conics S is the secant variety of V and has dimension 4.

To describe the $\mathrm{PGL}_3(\mathbb{C})$ -orbits with respect to the action induced by ψ , consider a point $p = [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \in \mathbb{P}^5$. Then p can either be mapped by an element of $\psi(\mathrm{PGL}_3(\mathbb{C}))$ to a point of the form $[a : 0 : 0 : b : 0 : 0]$, where $[a : b] \in \mathbb{P}^1$, or to the point $[1 : 0 : 0 : 0 : 0 : 1]$ and these points are all in different $\psi(\mathrm{PGL}_3(\mathbb{C}))$ -orbits. The stabilizer of $[1 : 0 : 0 : 0 : 0 : 1]$ in $\psi(\mathrm{PGL}_3(\mathbb{C}))$ is the subgroup of matrices of the form

$$\begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}, \text{ where } g \in \mathrm{PGL}_3(\mathbb{C}) \text{ has the form } \begin{bmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & c & 1 \end{bmatrix}.$$

Therefore, the orbit of $[1 : 0 : 0 : 0 : 0 : 1]$ under $\psi(\mathrm{PGL}_3(\mathbb{C}))$ has dimension 5. The orbit of a point of the form $[a : 0 : 0 : b : 0 : 0]$, on the other hand, has dimension 2. So we have a family parametrized by \mathbb{P}^1 of orbits of dimension 2 and one orbit of dimension 5. In particular, there is no $\psi(\mathrm{PGL}_3(\mathbb{C}))$ -invariant subset of dimension 4.

The following observation is easy but useful. We leave its proof to the reader.

Lemma 3.2. *Let X and Y be two projective varieties with biregular actions of a group G and let $f: X \dashrightarrow Y$ be a G -equivariant rational map. Then the indeterminacy locus $I_f \subset X$ and the exceptional divisor $\text{Exc}(f) \subset X$ are G -invariant sets.*

Note that Lemma 3.2 implies in particular that all equivariant rational maps with respect to actions without orbits of codimension ≥ 2 are morphisms.

Lemma 3.3. *Let M and M' be irreducible complex projective varieties such that $M \times \mathbb{P}^n$ and $M' \times \mathbb{P}^n$ are birationally equivalent. Then the standard embeddings*

$$\Psi: \text{PGL}_{n+1}(\mathbb{C}) \rightarrow \text{Bir}(\mathbb{P}^n \times M) \text{ and } \Psi': \text{PGL}_{n+1}(\mathbb{C}) \rightarrow \text{Bir}(\mathbb{P}^n \times M')$$

are conjugate if and only if M and M' are birationally equivalent.

Proof. If M and M' are birationally equivalent it follows directly that Ψ and Ψ' are conjugate. On the other hand, assume that there exists a birational map $A: \mathbb{P}^n \times M \dashrightarrow \mathbb{P}^n \times M'$ that conjugates Ψ to Ψ' , i.e. $A \circ \Psi(g) = \Psi'(g) \circ A$ for all $g \in \text{PGL}_{n+1}(\mathbb{C})$. The images $\Psi(\text{PGL}_{n+1}(\mathbb{C}))$ and $\Psi'(\text{PGL}_{n+1}(\mathbb{C}))$ permute the fibers $\{p\} \times M$, $p \in \mathbb{P}^n$ and $\{p\} \times M'$, $p \in \mathbb{P}^n$ respectively. By Lemma 3.2, no fiber is fully contained in the exceptional locus of A .

The fiber

$$F := [1 : 1 : \dots : 1] \times M \subset \mathbb{P}^n \times M$$

consists of all fixed points of the image of the subgroup of coordinate permutations $\Psi(\mathcal{S}_{n+1})$ and it is isomorphic to M . Correspondingly, the fiber

$$F' := [1 : 1 : \dots : 1] \times M' \subset \mathbb{P}^n \times M'$$

consists of all fixed points of $\Psi'(\mathcal{S}_{n+1})$ and is isomorphic to M' . Hence the strict transform of F under A is F' and we obtain that M and M' are birationally equivalent. \square

Proposition 3.4. *Let $\varphi, \psi: \text{PGL}_3(\mathbb{C}) \rightarrow \text{PGL}_6(\mathbb{C})$ be the homomorphisms defined in Section 3.1. The subgroups $\varphi(\text{PGL}_3(\mathbb{C}))$ and $\psi(\text{PGL}_3(\mathbb{C}))$ are not conjugate in Cr_5 .*

Proof. Assume that there is an element $f \in \text{Cr}_5$ conjugating $\varphi(\text{PGL}_3(\mathbb{C}))$ to $\psi(\text{PGL}_3(\mathbb{C}))$. Note that \mathbb{P}^5 has no $\psi(\text{PGL}_3(\mathbb{C}))$ -invariant subset of dimension 4. Hence, by Lemma 3.2, f must be a birational morphism and therefore an automorphism. But this isn't possible since the action of $\varphi(\text{PGL}_3(\mathbb{C}))$ has an orbit of dimension 4 and the action of $\psi(\text{PGL}_3(\mathbb{C}))$ does not. \square

3.2. A rational action on the space of plane conics. Our goal is to extend the group homomorphism $\varphi: \text{PGL}_3(\mathbb{C}) \rightarrow \text{PGL}_6(\mathbb{C})$ to a group homomorphism

$$\Phi: \text{Cr}_2 \rightarrow \text{Cr}_5.$$

A first naive idea is to check whether the map $\Psi: \{\text{PGL}_3(\mathbb{C}), \sigma\} \rightarrow \text{Cr}_5$ defined by $\Psi(g) = \varphi(g)$ for $g \in \text{PGL}_3(\mathbb{C})$ and $\Psi(\sigma) = [x_0^{-1} : x_2^{-1} : \dots : x_5^{-1}]$ extends to a group homomorphism $\text{Cr}_2 \rightarrow \text{Cr}_5$. However, $\Psi(\sigma)$ and $\Psi(h)$ don't satisfy relation (3) of Lemma A.9. Let $h = [Z - Z : Z - Y : Z] \in \text{Cr}_2$, then

$$([x_0^{-1} : x_1^{-1} : \dots : x_5^{-1}] \circ \varphi(h))^3 \neq \text{id}.$$

In [Giz99], Gizatullin constructs an extension $\Phi: \text{Cr}_2 \rightarrow \text{Cr}_5$ of φ , defined by $\Phi|_{\text{PGL}_3(\mathbb{C})} = \varphi$ and

$$\Phi(\sigma) = [x_1x_2 : x_0x_2 : x_0x_1 : x_3x_0 : x_4x_1 : x_5x_2].$$

He shows the following:

Proposition 3.5 ([Giz99]). *The map $\Phi: \text{Cr}_2 \rightarrow \text{Cr}_5$ is a group homomorphism.*

3.3. The dual action. We can also look at the representation $\varphi^\vee: \text{PGL}_3(\mathbb{C}) \rightarrow \text{PGL}_6(\mathbb{C})$ that is defined by

$$\varphi^\vee(g) := {}^t\varphi(g)^{-1}.$$

In other words, $\varphi^\vee = \varphi \circ \alpha$, where $\alpha: \text{PGL}_3(\mathbb{C}) \rightarrow \text{PGL}_3(\mathbb{C})$ is the algebraic automorphism $g \mapsto ({}^tg)^{-1}$.

Let $A = (a_{ij})$ be a 3×3 matrix. The cofactor matrix $C(A)$ of A is given by

$$C_{ij}(A) = (-1)^{i+j} A_{ij},$$

where A_{ij} is the i, j -minor of A , i.e. the determinant of the 2×2 -matrix obtained by removing the i -th row and j -th column of A . We denote by

$$\text{Ad}(A) := {}^tC(A)$$

the *adjugate* matrix of A . This is a classical construction and it is well known that $\text{Ad}(AB) = \text{Ad}(B)\text{Ad}(A)$ and that if A is invertible, then $\text{Ad}(A) = \det(A)A^{-1}$. In particular, $\text{Ad}: \mathbb{P}M_3 \dashrightarrow \mathbb{P}M_3$ is a birational map. The conic corresponding to the symmetric matrix A is the dual of the conic corresponding to the symmetric matrix A . This is one of the birational maps that A. R. Williams described 1938 in his paper “Birational transformations in 4-space and 5-space” ([Wil38]).

Lemma 3.6. *We identify \mathbb{P}^5 with the projectivized space of symmetric 3×3 matrices $\mathbb{P}M_3$. The birational transformation $\text{Ad} \in \text{Cr}_5$ is given by*

$$\text{Ad} := [x_1x_2 - x_3^2 : x_0x_2 - x_4^2 : x_0x_1 - x_5^2 : x_4x_5 - x_0x_3 : x_3x_5 - x_1x_4 : x_3x_4 - x_2x_5].$$

Moreover, ad conjugates φ to φ^\vee .

Proof. It is a straightforward calculation that the rational map Ad from \mathbb{P}^5 to itself that corresponds to Ad is given by

$$\text{Ad} := [x_1x_2 - x_3^2 : x_0x_2 - x_4^2 : x_0x_1 - x_5^2 : x_4x_5 - x_0x_3 : x_3x_5 - x_1x_4 : x_3x_4 - x_2x_5].$$

The actions of $\text{PGL}_3(\mathbb{C})$ on $\mathbb{P}M_3$ induced by φ and φ^\vee are given by $\varphi(g)(X) = gX({}^tg)$ and $\varphi^\vee(g)X = {}^t(g^{-1})Xg^{-1}$ respectively, for all $X \in \mathbb{P}M_3$. We obtain

$$\text{Ad}(\varphi(g)(X)) = \text{Ad}({}^tg)\text{Ad}(X)\text{Ad}(g) = ({}^tg)^{-1}\text{Ad}(X)g^{-1} = \varphi^\vee(g)\text{Ad}(X).$$

□

Remark 4. The Blow-up Q of \mathbb{P}^5 along the Veronese surface is the so called *space of complete conics*. Let $U \subset \mathbb{P}^5$ be the open orbit of the PGL_3 -action on \mathbb{P}^5 given by φ , i.e. $U = \mathbb{P}^5 \setminus S$. Then U can be embedded into $\mathbb{P}(\mathbb{C}^6) \times \mathbb{P}((\mathbb{C}^6)^\vee)$ by sending a conic $C \in U$ to the pair (C, C^\vee) , where \mathbb{C}^\vee denotes the dual conic of C . It turns out that Q is isomorphic to the closure of U in $\mathbb{P}(\mathbb{C}^6) \times \mathbb{P}((\mathbb{C}^6)^\vee)$. Moreover, the PGL_3 -action on \mathbb{P}^5 given by φ lifts to an algebraic action on Q and the birational map ad to an automorphism of Q . More details on this subject can be found for example in [Bri89].

Lemma 3.6 shows that the representations φ and φ^\vee are conjugate to each other in Cr_5 by the birational transformation Ad . By conjugating $\Phi(\sigma)$ with Ad we can extend φ^\vee to the *dual embedding* $\Phi^\vee: \text{Cr}_2 \rightarrow \text{Cr}_5$ and obtain

$$\begin{aligned} \Phi^\vee(\sigma) = & [(x_1x_2 - x_3^2)^2x_0 : (x_0x_2 - x_4^2)^2x_1 : (x_0x_1 - x_5^2)^2x_2 : \\ & (x_0x_2 - x_4^2)(x_0x_1 - x_5^2)x_3 : (x_1x_2 - x_3^2)(x_0x_1 - x_5^2)x_4 : (x_1x_2 - x_3^2)(x_0x_2 - x_4^2)x_5]. \end{aligned}$$

3.4. Geometry of Φ . The embedding Φ induces a rational action of Cr_2 on the space of conics on \mathbb{P}^2 . The action of $\Phi(\sigma)$ can be viewed geometrically as follows (compare with [Giz99, Introduction]): Let $Q_0 := [1 : 0 : 0]$, $Q_1 := [0 : 1 : 0]$ and $Q_2 := [0 : 0 : 1]$. Let $C \subset \mathbb{P}^2$ be a conic that doesn't pass through any of the points Q_i . Write

$$C = \{a_{00}X^2 + a_{11}Y^2 + a_{22}Z^2 + 2a_{12}YZ + 2a_{02}XZ + 2a_{01}XY = 0\} \subset \mathbb{P}^2$$

Denote by P_{1i}, P_{2i} the points of intersection of C with the lines l_i , where $l_0 := \{X = 0\}$, $l_1 := \{Y = 0\}$ and $l_2 := \{Z = 0\}$. Denote by f_{1i} and f_{2i} the lines passing through Q_i and P_{1i} respectively through Q_i and P_{2i} . The images $\sigma(f_{ji})$ are again lines passing through the points Q_i . Let P'_{1i} and P'_{2i} be the intersection points of $\sigma(f_{1i})$ and $\sigma(f_{2i})$ with l_i . One checks that the conic D defined by the equation

$$a_{11}a_{22}x_0^2 + a_{00}a_{22}x_1^2 + a_{00}a_{11}x_2^2 + 2a_{00}a_{12}x_1x_2 + 2a_{11}a_{02}x_0x_2 + 2a_{22}a_{01}x_0x_2 = 0$$

passes through the points P'_{ij} . Since no 4 of the 6 points P_{ij} lie on the same line, D is the unique conic through the points P_{ij} . We have thus proven the following:

Proposition 3.7. *For a general conic $C \subset \mathbb{P}^2$ there exists a unique conic D through the six points P'_{ij} and D is the image of C under $\Phi(\sigma)$.*

Notice as well that the indeterminacy points of $\Phi(\sigma)$ in \mathbb{P}^5 correspond to the subspace of dimension 2 of conics passing through the points Q_1, Q_2, Q_3 and the subspaces of dimension 2 of conics consisting of one l_i and any other line. The three subspaces of dimension 4 of conics passing through one of the points Q_i are contracted by the action of $\Phi(\sigma)$ and form the exceptional divisor.

In homogeneous coordinates of \mathbb{P}^5 , the four planes of indeterminacy locus of $\Phi(\sigma)$ can be described as follows

$$E_0 = \{x_1 = x_2 = x_3 = 0\}, E_1 = \{x_0 = x_2 = x_4 = 0\}, E_2 = \{x_0 = x_1 = x_5 = 0\}$$

$$\text{and } F = \{x_1 = x_2 = x_3 = 0\}.$$

The exceptional divisor of $\Phi(\sigma)$ consists of the three hyperplanes

$$H_0 = \{x_0 = 0\}, H_1 = \{x_1 = 0\}, H_2 = \{x_2 = 0\},$$

The hyperplanes H_0, H_1 and H_2 are contracted by $\Phi(\sigma)$ onto the planes E_0, E_1 and E_2 respectively. Note as well that E_0, E_1 and E_2 are contained in the secant variety $S \subset \mathbb{P}^5$ of the Veronese surface V and they are tangent to V .

The geometrical description of the rational action of $\Phi^\vee(\sigma)$ on the space of conics is the dual of the construction described above. If C is a conic not passing through any of the points Q_0, Q_1, Q_2 , we get $\Phi^\vee(\sigma)(C)$ in the following way: let $l_{i,1}, l_{i,2}$ be the tangents of C passing through the point Q_i . Then the images of the $l_{i,1}$ and $l_{i,2}$ under σ are lines again. There exists a unique conic having all the lines $\sigma(l_{i,1})$ and $\sigma(l_{i,2})$ for all i as tangents.

These geometrical constructions show that $\Phi(\text{Cr}_2)$ preserves the space of conics consisting of double lines and therefore the Veronese surface V in \mathbb{P}^5 . The injective morphism

$$v: \mathbb{P}^2 \rightarrow \mathbb{P}^5, [X : Y : Z] \mapsto [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$$

is called the *Veronese morphism*. It is an isomorphism onto its image, which is V . It is well known that v is $\text{PGL}_3(\mathbb{C})$ -equivariant with respect to the standard action and the action induced by Φ respectively. The restriction of $\Phi(\sigma)$ to V is a birational transformation. We therefore obtain a rational action of Cr_2 on $V \simeq \mathbb{P}^2$. Since the restriction of this rational action to $\text{PGL}_3(\mathbb{C})$ is the standard action, we obtain by Corollary A.12 that v is Cr_2 -equivariant.

We observe as well that $\Phi(\text{Cr}_2)$ preserves the secant variety $S \subset \mathbb{P}^5$ of V . Note that S is the image of the morphism:

$$\mathcal{S}: \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow S \subset \mathbb{P}^5,$$

that maps the point $[X : Y : Z], [U : V : W] \in \mathbb{P}^2 \times \mathbb{P}^2$ to the point

$$[XU : YV : ZW : 1/2(YW + UZ) : 1/2(XW + ZU) : 1/2(XV + YU)].$$

Note that s is generically 2 : 1. Again, the geometrical construction above shows that s is Cr_2 -equivariant with respect to the diagonal action on $\mathbb{P}^2 \times \mathbb{P}^2$ and the action given by Φ on \mathbb{P}^5 respectively.

We obtain the following sequence of Cr_2 -equivariant maps:

$$\mathbb{P}^2 \xrightarrow{\Delta} \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{s} \mathbb{P}^5,$$

where Δ is the diagonal embedding. This proves part (2) to (4) of Theorem 1.4.

The observation that $\Phi(\text{Cr}_2)$ preserves the Veronese surface and extends the canonical rational action of Cr_2 has a nice consequence:

Proposition 3.8. *Let $f \in \text{Cr}_2$. Then $\deg(f) \leq \deg(\Phi(f))$.*

Proof. Denote by $v: \mathbb{P}^2 \rightarrow \mathbb{P}^5$ the Veronese embedding. Let $C \subset \mathbb{P}^2$ be a general conic. The image $v(C) \subset \mathbb{P}^5$ is a curve of degree 4 given by the intersection of a hyperplane $H \subset \mathbb{P}^5$ and the Veronese surface. Let $f \in \text{Cr}_2$ be a birational transformation of degree d . The strict transform $f(C')$ of a general conic $C' \subset \mathbb{P}^2$ intersects C in $4d$ different points. So $v(C)$ intersects $v(f(C'))$ in $4d$ different points. By the above results we know that $v(f(C')) = \Phi(f)(v(C'))$. Let $d' = \deg(\Phi(f))$. Since $v(C')$ is a curve of degree 4, this yields that $\Phi(f)(v(C'))$ is a curve of degree $4d'$. The curve $\Phi(f)(v(C'))$ intersects the hyperplane H in $4d$ points, hence $d' \geq d$. \square

3.5. Two induced embeddings from Cr_2 into Cr_4 . The birational map $Ad \in \text{Cr}_5$ contracts the secant variety $S \subset \mathbb{P}^5$ onto the Veronese surface $V \subset \mathbb{P}^5$. However, the exceptional locus of $\Psi^\vee(\sigma) = Ad\Phi(\sigma)Ad$ consists of the three hyperplanes

$$G_0 = \{z_1 z_2 - z_3^2 = 0\}, G_1 = \{z_0 z_2 - z_4^2 = 0\}, G_2 = \{z_0 z_1 - z_5^2 = 0\},$$

with respect to homogeneous coordinates $[z_0 : z_1 : z_2 : z_3 : z_4 : z_5]$ of \mathbb{P}^5 .

This implies in particular that the restriction of $\Phi^\vee(\sigma)$ to S induces a birational map of S and therefore that any element in $\Phi^\vee(\text{Cr}_2)$ restricts to a birational map of S .

Since S is a cubic hypersurface and contains the two disjoint planes

$$E_1 = \{z_1 = z_2 = z_3 = 0\}, E_2 = \{z_0 = z_4 = z_5 = 0\},$$

it is rational. Explicitly, projection onto E_1 and E_2 yields the birational map $A: S \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \mapsto [z_1 : z_2 : z_3], [z_0 : z_4 : z_5].$$

The inverse transformation A^{-1} is given by

$$[x_0 : x_1 : x_2], [y_0 : y_1 : y_2] \mapsto [p_2 y_0, p_1 x_0, p_1 x_1, p_1 x_2, p_2 y_1, p_2 y_2],$$

where $p_1 = (x_0 y_1^2 + x_1 y_2^2 - 2x_2 y_1 y_2)$ and $p_2 = y_0(x_0 x_1 - x_2^2)$.

Let $f \in \text{Cr}_2$. As seen above, both images $\Phi(f)$ and $\Phi^\vee(f)$ restrict to a birational map of S . So conjugation of Φ and Φ^\vee by A yields two embeddings from Cr_2 into $\text{Bir}(\mathbb{P}^2 \times \mathbb{P}^2) \simeq \text{Cr}_4$, which we denote by Ψ_1 and Ψ_2 respectively.

Proof of Proposition 1.5. Irreducibility is proved in Section 3.8.

By Theorem 2.5, all tori $D_2 \subset \text{Cr}_4$ are conjugate to the standard torus $D_2 \subset \text{Cr}_4$. We calculate the map that conjugates $\Psi_1(D_2) = \Psi_2(D_2)$ to the image of the standard embedding of D_2 explicitly. Let $\rho: \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be the birational transformation defined by

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_2 y_0 : x_0 y_1 : x_2 y_1], [x_0 y_1^2 : x_1 y_2^2 : x_2 y_1 y_2]).$$

The inverse map ρ^{-1} is given by

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_1^2 y_2^2 : x_2^2 y_0 y_1 : x_1 x_2 y_2^2], [x_0 y_0 : x_2 y_0 : x_1 y_2]).$$

One calculates that $\rho A \Psi_1([aX : bY : cZ]) A^{-1} \rho^{-1}$ maps $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$ to $([ax_0 : bx_1 : cx_2], [y_0 : y_1 : y_2])$. Correspondingly, $\rho A \Psi_2([aX : bY : cZ]) A^{-1} \rho^{-1}$ maps $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$ to $([a^{-1}x_0 : b^{-1}x_1 : c^{-1}x_2], [y_0 : y_1 : y_2])$. So the second coordinates parametrize the closures of the D_2 -orbits. Since W_2 normalizes D_2 , its image preserves the D_2 -orbits. We thus obtain two homomorphisms

$$\chi_1: W_2 \rightarrow \text{Cr}_2, \chi_2: W_2 \rightarrow \text{Cr}_2$$

by just considering the rational action of W_2 on the second coordinate.

Assume that there exists an element $A \in \text{Bir}(\mathbb{P}^2 \times \mathbb{P}^2)$ that conjugates Ψ to Ψ^\vee . As A normalizes $\Psi_1(D_2) = \Psi_2(D_2)$, it preserves the $\Psi_1(D_2)$ -orbits as well. Hence by restriction on the second coordinate, it conjugates χ_1 to χ_2 . It therefore suffices to show that χ_1 and χ_2 are not conjugate.

In Cr_2 we have

$$f := [XY : YZ : Z^2] = \tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2,$$

where $\tau_1 = [Z : Y : X]$, $\tau_2 = [Y : Z : X]$ and $g_0 = [Y - X : Y : Z]$. By calculating the corresponding images under Φ we obtain

$$\Phi(f) = \Phi(\tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2) = [x_0 x_1 : x_1 x_2 : x_2^2 : x_2 x_3 : -x_2 x_5 + 2x_3 x_4 : x_1 x_4]$$

and $\Phi^\vee(f) = [g_0 : g_1 : g_2 : g_3 : g_4 : g_5]$, where

$$\begin{aligned} g_0 &= (x_0 x_1 - x_5^2)^2 x_0, \\ g_1 &= x_0^2 x_1^2 x_2 - 2x_0 x_1 x_2 x_5^2 - 4x_0 x_1 x_3 x_4 x_5 + 4x_0 x_3^2 x_5^2 + 4x_1 x_4^2 x_5^2 + x_2 x_5^4 - 4x_3 x_4 x_5^3, \\ g_2 &= (x_0 x_2 - x_4^2)^2 x_1, \\ g_3 &= (x_0 x_2 - x_4^2)(x_0 x_1 x_3 - 2x_1 x_4 x_5 + x_3 x_5^2), \\ g_4 &= -(x_0 x_2 - x_4^2)(x_0 x_1 - x_5^2) x_5, \end{aligned}$$

$$g_5 = (x_0x_1 - x_5^2)(x_0x_1x_4 - 2x_0x_3x_5 + x_4x_5^2).$$

This yields

$$\chi_1(f) = [(y_1 - 2y_2)^2 : y_0y_1 : -y_2(y_1 - 2y_2)]$$

and

$$\begin{aligned} \chi_2(f) = [y_0^2y_1 + 4y_0y_1^2 - 6y_0y_1y_2 - 3y_1y_2^2 + 4y_2^3 : y_0(y_0 + 2y_1 - 3y_2)^2 : \\ (2y_0y_1 - y_0y_2 - y_2^2)(y_0 + 2y_1 - 3y_2)]. \end{aligned}$$

We show that these two transformations are not conjugate in Cr_2 . With respect to affine coordinates $[y_0 : y_1 : 1]$ one calculates

$$\chi_1(f)^2 = \left(\frac{y_0y_1 - 2y_1 + 4}{y_1 - 2}, y_1 \right).$$

From this we see that the integer sequence $\deg(\chi_1(f)^n)$ grows linearly in n and is, in particular, not bounded.

Let $A = [y_0 - y_2 : y_1 - y_2 : y_2]$. Then

$$A\chi_2(f)^2A^{-1} = [-y_0^2y_1^2(2y_1 + y_0) : y_0^2y_1^2(3y_1 + 2y_0) : p(y_0, y_1, y_2)(3y_1 + 2y_0)(2y_1 + y_0)],$$

where $p(y_0, y_1, y_2) = (6y_1^2y_2 + 7y_2y_0y_1 + 6y_0y_1^2 + 2y_0^2y_2 + 2y_0^2y_1)$. We claim that

$$f_A^n = A\chi_2(f)^{2n}A^{-1} = [-y_0^2y_1^2(2ny_1 + (2n-1)y_0) : y_0^2y_1^2((2n+1)y_1 + 2ny_0) : f_n],$$

where $f_n = (2ny_1 + (2n-1)y_0)((2n+1)y_1 + 2ny_0)p_n(y_0, y_1, y_2)$ for some homogeneous $p_n \in \mathbb{C}[y_0, y_1, y_2]$ of degree 3. Note that this claim implies in particular that $\deg(\chi_2(f)^n)$ is bounded for all n and hence that $\chi_1(f)$ and $\chi_2(f)$ are not conjugate.

To prove the claim we proceed by induction. Assume that f_A^n has the desired form. One calculates that the first coordinate of $f_A^{n+1} = A\chi_2(f)^2A^{-1} \circ f_A^n$ is

$$-ry_0^2y_1^2((2n+2)y_1 + (2n+1)y_0),$$

the second coordinate is

$$ry_0^2y_1^2((2n+3)y_1 + (2n+1)y_0)$$

and the third coordinate

$$r((2n+2)y_1 + (2n+1)y_0)((2n+3)y_1 + (2n+1)y_0)p_{n+1}(x_0, x_1, x_2),$$

where $r = y_0^4y_1^4(2ny_0 + (2n-1)y_1)^2((2n+1)y_0 + 2ny_1)^2$ and $p_n \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree 3. This proves the claim. \square

3.6. A volume form. Let M be a complex projective manifold. It is sometimes interesting to study subgroups of $\text{Bir}(M)$ that preserve a given form. In [Bla13] and [DL16] the authors study for example birational maps of surfaces that preserve a meromorphic symplectic form (see [CK15] for the 3-dimensional case). In [Giz08] and [CD16] Cremona transformations in dimension 3 preserving a contact form are studied.

Define

$$F := \det \begin{pmatrix} x_0 & x_5 & x_4 \\ x_5 & x_1 & x_3 \\ x_4 & x_3 & x_2 \end{pmatrix}$$

and let

$$\Omega := \frac{x_5^6}{F^2} \cdot dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

Then Ω is a 5-form on \mathbb{P}^5 with a double pole along the secant variety of the Veronese surface. Note that the total volume of \mathbb{P}^5 is infinite.

Proposition 3.9. *All elements in $\Phi(\text{Cr}_2)$ preserve Ω .*

Proof. We show that $\Phi(\text{PGL}_3(\mathbb{C}))$ and $\Phi(\sigma)$ preserve Ω .

Let $g = [-X : -Y : Z] \in \Phi(\text{PGL}_3(\mathbb{C}))$. One checks that $\Phi(g)$ preserves Ω . Since $\Phi(\text{PGL}_3(\mathbb{C}))$ preserves F , we have that $\Phi(fgf^{-1})$ preserves Ω as well. As $\Phi(\text{PGL}_3(\mathbb{C}))$ is simple, the whole group preserves Ω .

With respect to affine coordinates given by $x_5 = 1$, we have

$$\Phi(\sigma) = (x_1, x_0, x_0x_1x_2^{-1}, x_0x_3x_2^{-1}, x_1x_4x_2^{-1}).$$

A direct calculation yields $\Omega \circ \Phi(\sigma) = \Omega$. \square

3.7. Polynomial automorphisms. In this section we will prove Claim (6) of Theorem 1.4 as well as Theorem 1.6. Let $\text{Aut}(\mathbb{A}^2) \subset \text{Cr}_2$ be the subgroup of automorphisms of the affine plane with respect to the affine coordinates $[1 : X : Y]$. By the theorem of Jung and van der Kulk (see for example [Lam02]), $\text{Aut}(\mathbb{A}^2)$ has the following amalgamated product structure

$$\text{Aut}(\mathbb{A}^2) = \text{Aff}_2 *_{\cap} \mathcal{J}_2,$$

where \mathcal{J}_2 denotes the subgroup of elementary automorphisms, which is the subgroup of all elements of the form

$$\{(c_1X, c_2Y + p(X)) \mid c_1, c_2 \in \mathbb{C}, p(X) \in \mathbb{C}[X]\}.$$

Let $f \in \text{Aut}(\mathbb{A}^2)$ and assume that $f = a_1j_1a_2j_2 \cdots j_{n-1}a_n$, where $a_i \in \text{Aff}_2$ and $j_i \in \mathcal{J}_2 \setminus \text{Aff}_2$. It is well known that $\deg(f) = \deg(j_1)\deg(j_2) \cdots \deg(j_{n-1})$.

Let $\text{Aut}(\mathbb{A}^5) \subset \text{Cr}_5$ be given by the affine coordinates $[1 : x_1 : \cdots : x_5]$. Lemma 3.10 follows from a direct calculation.

Lemma 3.10. *The image $\Phi(\text{Aff}_2)$ is contained in Aff_5 .*

We consider the following elements in \mathcal{J}_2 :

$$f_n^\lambda := (X, Y + \lambda X^n),$$

where $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{C}$.

Lemma 3.11. *For all $n \in \mathbb{Z}_{\geq 0}$ we have*

$$\Phi(f_n^\lambda) = (x_1, x_2 + \lambda^2 x_1^n + \lambda x_3 A_n - \lambda x_4 x_1 A_{n-1}, x_3 + \lambda x_1 B_{n-1}, x_4 + \lambda B_n, x_5),$$

where

$$A_n = 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k+1} x_5^{n-2k-1} (x_5^2 - x_1)^k$$

and

$$B_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x_5^{n-2k} (x_5^2 - x_1)^k.$$

Moreover, the following recursive identities hold:

$$A_n = 2x_5 A_{n-1} - x_1 A_{n-2},$$

$$B_n = 2x_5 B_{n-1} - x_1 B_{n-2}.$$

Proof. For $n = 0$ and $n = 1$ the claim follows from a direct calculation.

Let $s := (X, XY) \in \text{Cr}_2$. Then we have $f_{n+1}^\lambda = s f_n^\lambda s^{-1}$. In Cr_2 the identity $s = \tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2$ holds, where $\tau_1 = (XY^{-1}, Y^{-1})$, $\tau_2 = (Y^{-1}, XY^{-1})$ and $g_0 = (X, X_Y)$. Note that τ_1 and τ_2 are elements of PGL_3 . If we calculate the corresponding images under Φ we obtain

$$\Phi(s) = \Phi(\tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2) = (x_1, x_1 x_2, x_1 x_4, 2x_4 x_5 - x_3, x_5)$$

and

$$\Phi(s^{-1}) = (x_1, x_2 x_1^{-1}, 2x_3 x_5 x_1^{-1} - x_4, x_3 x_1^{-1}, x_5).$$

One calculates

$$\begin{aligned} s f_n^\lambda s^{-1} &= (x_1, x_2 + \lambda^2 x_1^{n+1} + \lambda x_3(2x_5 - x_1)A_{n-1} - \lambda x_4 x_1 A_n, x_3 + \lambda x_1 B_n, \\ &\quad x_4 - \lambda(2x_5 B_n - x_1 B_{n-1})). \end{aligned}$$

This shows by induction that

$$\Phi(f_n^\lambda) = (x_1, x_2 + \lambda^2 x_1^n + \lambda x_3 A_n - \lambda x_4 x_1 A_{n-1}, x_3 + \lambda x_1 B_{n-1}, x_4 + \lambda B_n, x_5),$$

where

$$\begin{aligned} A_n &= 2x_5 A_{n-1} - x_1 A_{n-2}, A_0 = 0, A_1 = 2; \\ B_n &= 2x_5 B_{n-1} - x_1 B_{n-2}, B_0 = 1, B_1 = x_5. \end{aligned}$$

These recursive formulas have the following closed form:

$$\begin{aligned} A_n &= \frac{(x_5 + \sqrt{x_5^2 - x_1})^n - (x_5 - \sqrt{x_5^2 - x_1})^n}{\sqrt{x_5^2 - x_1}}, \\ B_n &= \frac{1}{2} \left((x_5 - \sqrt{x_5^2 - x_1})^n + 1/2 \left(x_5 + \sqrt{x_5^2 - x_1} \right)^n \right). \end{aligned}$$

The claim follows. \square

Since Aff_n together with all the elements f_n^λ , $n \in \mathbb{Z}^+$, $\lambda \neq 0$ generates $\text{Aut}(\mathbb{A}^2)$, Lemma 3.11 shows that $\Phi(\text{Aut}(\mathbb{A}^2))$ is contained in $\text{Aut}(\mathbb{A}^5)$ and thus claim (6) of Theorem 1.4.

Lemma 3.12. *Let n and m be positive integers and A_n, B_m as in Lemma 3.11. Then*

$$A_n B_{m-1} - A_{n-1} B_m = P(x_1, x_5),$$

where $P \in \mathbb{C}[x_1, x_5]$ is a polynomial of degree $< \max\{m, n\}$.

Proof. If $n = 1$ or $m = 1$ the claim is true, since $A_0 = 0, A_1 = 2, B_0 = 1, B_1 = x_5$ and $\deg(A_k) = k - 1, \deg(B_k) = k$. By the identities from Lemma 3.11, one obtains

$$\begin{aligned} A_n B_{m-1} - A_{n-1} B_m &= (2x_5 A_{n-1} - x_1 A_{n-2}) B_{m-1} - A_{n-1} (2x_5 B_{m-1} - x_1 B_{m-2}) \\ &= x_1 (A_{n-1} B_{m-2} - A_{n-2} B_{m-1}). \end{aligned}$$

The claim follows by induction on m and n . \square

Lemma 3.13. *Let*

$$f = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_n^{\lambda_n}, \text{ where } \lambda_n \neq 0.$$

Then

$$\Phi(f) = (x_1, x_2 + F, x_3 + p_3(x_1, x_5) + \lambda_n x_1 B_{n-1}, x_4 + p_4(x_1, x_5) + \lambda_n B_n, x_5),$$

where $F = p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \dots + \lambda_n A_n) - x_4 x_1(\lambda_1 A_{n-1} + \dots + \lambda_n A_n)$ and $p_2, p_3, p_4 \in \mathbb{C}[x_1, x_5]$ are polynomials of degree $\leq n$. In particular, $\deg(\Phi(f)) = \deg(f)$.

Proof. It is easy to see that the third and fourth coordinate of $\Phi(f)$ have the claimed form. The more difficult part is the second coordinate.

For $n = 1$ the claim follows directly from Lemma 3.11. We proceed now by induction. Let $\lambda_{n+1} \neq 0$ and m be the largest number, such that $m \leq n$ and $n \neq 0$. By the induction hypothesis we may assume that the second coordinate of $\Phi(f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_m^{\lambda_m})$ has the form

$$x_2 + p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \cdots + \lambda_m A_m) - x_4 x_1(\lambda_1 A_0 + \cdots + \lambda_m A_{m-1}).$$

The second coordinate of $\Phi(f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_m^{\lambda_m}) \circ \Phi(f_n^{\lambda_n})$ is therefore

$$x_2 + p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \cdots + \lambda_m A_m + \lambda_n A_n) - x_4 x_1(\lambda_1 A_0 + \cdots + \lambda_m A_{m-1} + \lambda_n A_n) +$$

$$x_1 \sum_{k=1}^m \lambda_k (A_k B_{n-1} - A_{k-1} B_n).$$

By Lemma 3.12, $x_1 \sum_{k=1}^m \lambda_k (A_k B_{n-1} - A_{k-1} B_n)$ is a polynomial in x_1 and x_5 of degree $\leq n$. \square

Proof of Theorem 1.6. The first claim was proved in Proposition 3.8.

For the second part it suffices by the remark above on the amalgamated product structure to show that $\deg(\Phi(f)) = \deg(f)$ for all elements $f \in \mathcal{J}_2$. Composition with an element in Aff_2 doesn't change the degree. So it is enough to consider elements in \mathcal{J}_2 of the form $f = (X, Y + P(X))$, $P \in \mathbb{C}[X]$. For suitable $\lambda_i \in \mathbb{C}$ we have $f = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_n^{\lambda_n}$, where $\lambda_n \neq 0$. In Lemma 3.13 we've seen that Φ preserves the degree of these elements. \square

3.8. Irreducibility of Φ , Ψ_1 and Ψ_2 . First we show that Φ is irreducible. Assume that there is a rational dominant map $\pi: \mathbb{P}^5 \dashrightarrow M$ to a variety M with an algebraic embedding $\varphi_M: \text{Cr}_2 \rightarrow \text{Bir}(M)$ such that A is Cr_2 -equivariant. Since φ_M is algebraic, we may assume that $\text{PGL}_3(\mathbb{C})$ acts regularly on M . We obtain that the restriction of A to the open $\text{PGL}_3(\mathbb{C})$ -invariant subset $U \subset \mathbb{P}^5$ consisting of all smooth conics is a $\text{PGL}_3(\mathbb{C})$ -equivariant morphism, whose image is an open dense subset of M on which $\text{PGL}_3(\mathbb{C})$ acts transitively. Note that this implies $\dim(M) > 1$.

If $\dim M = 2$, we obtain by Theorem 1.3 that $M \simeq \mathbb{P}^2$ with the standard action of $\text{PGL}_3(\mathbb{C})$. The stabilizer in $\text{PGL}_3(\mathbb{C})$ of a point in $U \subset \mathbb{P}^5$ is isomorphic to $\text{SO}_3(\mathbb{C})$. On the other hand the stabilizer in $\text{PGL}_3(\mathbb{C})$ of a point in \mathbb{P}^2 is isomorphic to the group of affine transformations $\text{Aff}_2 = \text{GL}_2(\mathbb{C}) \ltimes \mathbb{C}^2$. Since $\text{SO}_3(\mathbb{C})$ can not be embedded into Aff_2 , the case $\dim(M) = 2$ is not possible.

If $\dim(M) = 3$, we find, by Theorem 4.1, a $\text{PGL}_3(\mathbb{C})$ -equivariant projection $M \dashrightarrow \mathbb{P}^2$ and are again in the case $\dim(M) = 2$.

If $\dim(M) = 4$, let $p \in M$ be a general point and $F_p := A^{-1}(p) \subset \mathbb{P}^5$ the fiber of A . Let $q \in F_p$ be a point that is only contained in one connected component C of F_p . Again, the stabilizer of q is isomorphic to $\text{SO}_3(\mathbb{C})$. This implies that $\text{SO}_3(\mathbb{C})$ acts regularly on the curve C with a fixpoint. The group of birational transformations of C is isomorphic to $\text{PGL}_2(\mathbb{C})$, is abelian or is finite. In all cases we obtain that the connected component of the identity $\text{SO}_3(\mathbb{C})^0$ fixes C pointwise. In other words, the group $\text{SO}_3(\mathbb{C})^0$ preserves each conic of the family of conics in \mathbb{P}^2 parametrized by C . This is not possible.

The proof that Ψ_1 and Ψ_2 are irreducible is done analogously.

4. $\mathrm{PGL}_{n+1}(\mathbb{C})$ -ACTIONS IN CODIMENSION 1

In this section we look at algebraic embeddings of $\mathrm{PGL}_{n+1}(\mathbb{C})$ into $\mathrm{Bir}(M)$ for complex projective varieties M of dimension $n+1$. Our aim is to prove Theorem 4.1.

Theorem 4.1. *Let $n \geq 2$ and let M be a smooth projective variety of dimension $n+1$ with a non-trivial $\mathrm{PGL}_{n+1}(\mathbb{C})$ -action. Then, up to birational conjugation and automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$, we have one of the following:*

- (1) $M \simeq F_l = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-l(n+1)))$ for a unique element $l \in \mathbb{Z}_{\geq 0}$ and $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts as in Example 1.8.
- (2) $M \simeq \mathbb{P}^n \times C$ for a unique smooth curve C and $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts on the first factor as in Example 1.1.
- (3) $M \simeq \mathbb{P}(T\mathbb{P}^2)$ and PGL_3 acts as in Example 1.9.
- (4) $M \simeq \mathbb{G}(1, 3)$ and $\mathrm{PGL}_4(\mathbb{C})$ acts as in Example 1.10.

Moreover, these actions are not birationally conjugate to each other.

Remark 5. If M is rational and of dimension 2 or 3, this result can be deduced directly from the classification of maximal algebraic subgroups of Cr_n by Enriques, Umemura and Blanc ([Emr93], [Ume82b], [Ume85], [Ume82a], [Bla09]).

4.1. Classification of varieties and groups of automorphisms. With some geometric invariant theory and using results of Freudenthal about topological ends, the following classification can be made (see [CZ12] and the references in there):

Theorem 4.2. *Let M be a smooth projective variety of dimension $n+1$ with an action of $\mathrm{PGL}_{n+1}(\mathbb{C})$, where $n \geq 2$. Then we are in one of the following cases:*

- (1) $M \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))$ for some $k \in \mathbb{Z}_{\geq 0}$.
- (2) $M \simeq \mathbb{P}^n \times C$ for a curve C of genus ≥ 1 .
- (3) $M \simeq \mathbb{P}(T\mathbb{P}^2) \simeq \mathrm{PGL}_3(\mathbb{C})/B$, where $P \subset \mathrm{PGL}_3(\mathbb{C})$ is a maximal Borel subgroup.
- (4) $M \simeq \mathbb{G}(1, 3) \simeq \mathrm{PGL}_4(\mathbb{C})/P$, where $P \subset \mathrm{PGL}_4(\mathbb{C})$ is the parabolic subgroup consisting of matrices of the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

The connected components $\mathrm{Aut}^0(M)$ of the automorphism groups of the varieties M that appear in Theorem 4.2 are well known. Proofs of the following Proposition can be found in [Akh95].

Proposition 4.3. *We have*

- $\mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))) \simeq (\mathrm{GL}_{n+1}(\mathbb{C})/\mu_k) \rtimes \mathbb{C}[x_0, \dots, x_n]_k$, where $\mathbb{C}[x_0, \dots, x_n]_k$ denotes the additive group of homogeneous polynomials of degree k and $\mu_k \subset \mathbb{C}^*$ the group of all elements $c \in \mathbb{C}^*$ satisfying $c^k = 1$,
- $\mathrm{Aut}^0(\mathbb{P}^n \times C) \simeq \mathrm{PGL}_{n+1}(\mathbb{C}) \times \mathrm{Aut}^0(C)$,
- $\mathrm{Aut}^0(\mathbb{P}(T\mathbb{P}^2)) \simeq \mathrm{PGL}_3(\mathbb{C})$,
- $\mathrm{Aut}^0(\mathbb{G}(1, 3)) \simeq \mathrm{PGL}_4(\mathbb{C})$.

To describe the $\mathrm{PGL}_{n+1}(\mathbb{C})$ -actions on these varieties we recall some results about group cohomology.

4.2. Group cohomology. Let H be a group that acts by automorphisms on a group N . A *cocycle* is a map $\tau: H \rightarrow N$ such that $\tau(gh) = \tau(g)(g \cdot \tau(h))$ for all $g, h \in H$. Two cocycles τ and ν are *cohomologous* if there exists an $a \in N$ such that

$$\tau(g) = a^{-1}\nu(g)(g \cdot a) \text{ for all } g \in H.$$

The set of cocycles up to cohomology will be denoted by $H^1(H, N)$. If H acts trivially on N , the set $H^1(H, N)$ corresponds to the set of group homomorphisms $H \rightarrow N$. The following lemma is well known.

Lemma 4.4. *Let $G := N \rtimes H$ be a semi direct product of groups and $\pi: G \rightarrow H$ the canonical projection on H . Then there exists a bijection between $H^1(H, N)$ and the sections of π up to conjugation in N .*

There always exists the trivial cocycle $\tau_0: H \rightarrow N, g \mapsto e_N$. The set $H^1(G, N)$ is therefore a pointed set with basepoint τ_0 . Assume that G acts on two groups A and B by automorphisms. A G -homomorphism $\phi: A \rightarrow B$ induces a homomorphism of pointed sets

$$\phi_*: H^1(G, A) \rightarrow H^1(G, B)$$

given by $\phi_*(\tau) = \phi \circ \tau$.

Proposition 4.5 ([Ser79], p. 125, Proposition 1). *Let G be a group that acts by automorphisms on groups A, B and C . Every exact sequence of G -homomorphisms*

$$1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$$

induces an exact sequence of pointed sets

$$H^1(G, A) \rightarrow H^1(G, B) \rightarrow H^1(G, C).$$

4.3. Proof of Theorem 4.1.

4.3.1. Uniqueness of the actions. Now we show that $\mathrm{PGL}_{n+1}(\mathbb{C})$ can only be embedded into $\mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)))$ if and only if $n \mid k$. Then we show that in this case, up to conjugation and algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$, the embedding is unique.

By Proposition 4.3, $\mathrm{Aut}^0(\mathbb{P}(T\mathbb{P}^2)) \simeq \mathrm{PGL}_3(\mathbb{C})$ and $\mathrm{Aut}(\mathbb{G}(1, 3)) \simeq \mathrm{PGL}_4(\mathbb{C})$. The uniqueness of the embedding is clear in this cases since $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a simple group. If $M \simeq \mathbb{P}^n \times C$ uniqueness follows directly from the fact that $\mathrm{PGL}_{n+1}(\mathbb{C})$ does not embed into $\mathrm{Aut}(C)$.

Lemma 4.6. *A non-trivial group homomorphism $\mathrm{PGL}_n(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})/\mu_k$ exists if and only if $n \mid k$, where*

$$\mu_k = \{\lambda \mathrm{id} \mid \lambda \in \mathbb{C}, \lambda^k = 1\}.$$

Let n and k be positive integers such that $(n+1) \mid k$. Denote by $\mathbb{C}[x_0, \dots, x_n]_k$ the vector space of homogeneous polynomials of degree k . We define

$$G := \mathbb{C}[x_0, \dots, x_n]_k \rtimes \mathrm{PGL}_{n+1}(\mathbb{C}),$$

where the semi direct product is taken with respect to the action $g \cdot p = p \circ g^{-1}$. Here we look at $\mathrm{PGL}_{n+1}(\mathbb{C}) \subset \mathrm{GL}_{n+1}(\mathbb{C})/\mu_k$ as described in Lemma 4.6. Let $\pi: G \rightarrow \mathrm{PGL}_{n+1}(\mathbb{C})$ be the standard projection.

Lemma 4.7. *Up to conjugation, there exists a unique section $\iota: \mathrm{PGL}_{n+1}(\mathbb{C}) \rightarrow G$ of π .*

The results in Lemma 4.6 and Lemma 4.7 are certainly well known. A detailed proof can be found in [Ureon].

Lemma 4.8. $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts non-trivially on the fibration $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))$ with basis \mathbb{P}^n if and only if $k = l(n+1)$ for some nonnegative l . Moreover, in this case the action is unique up to conjugation and up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$.

Proof. Let $\phi: \mathrm{PGL}_{n+1}(\mathbb{C}) \rightarrow \mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)))$ be an algebraic embedding. By Proposition 4.3, there exists an exact sequence of algebraic homomorphisms

$$1 \rightarrow \mathbb{C}[x_0, \dots, x_n]_k \rightarrow \mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))) \rightarrow \mathrm{GL}_{n+1}(\mathbb{C})/\mu_k \rightarrow 1.$$

If ϕ is non-trivial, this induces a non-trivial algebraic homomorphism from $\mathrm{PGL}_{n+1}(\mathbb{C})$ into $\mathrm{GL}_{n+1}(\mathbb{C})/\mu_k$ and by Lemma 4.6 this is possible if and only if $(n+1) \mid k$. So assume that $k = l(n+1)$ for an integer l . It remains to show that in this case ϕ is unique up to conjugation and up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$. Let

$$F_l := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)).$$

We look at F_l as a \mathbb{P}^1 -fibration over the basis \mathbb{P}^n . So there is an exact sequence

$$1 \rightarrow \mathrm{Aut}_{\mathbb{P}^n}^0(F_l) \rightarrow \mathrm{Aut}^0(F_l) \xrightarrow{\pi} \mathrm{PGL}_n(\mathbb{C}) \rightarrow 1.$$

Here, $\mathrm{Aut}_{\mathbb{P}^n}^0(F_l) \simeq \mathbb{C}^* \ltimes \mathbb{C}[x_0, \dots, x_n]_k$ denotes the subgroup of automorphisms of F_l that fix the basis \mathbb{P}^n pointwise.

Let $H := \mathrm{PGL}_{n+1}(\mathbb{C})$. By Lemma 4.4, the sections of π up to conjugation are in bijection with

$$H^1(H, \mathrm{Aut}_{\mathbb{P}^n}^0(F_l)) = H^1(H, \mathbb{C}[x_0, \dots, x_n]_k \rtimes \mathbb{C}^* / \mu_k).$$

By Proposition 4.5, there is an exact sequence of pointed sets

$$H^1(H, \mathbb{C}[x_0, \dots, x_n]_k) \rightarrow H^1(H, \mathrm{Aut}_{\mathbb{P}^n}(F_l)) \rightarrow H^1(H, \mathbb{C}^* / \mu_k).$$

The action of H on \mathbb{C}^* / μ_k is trivial, so $H^1(H, \mathbb{C}^* / \mu_k)$ is the set of homomorphisms $H \rightarrow \mathbb{C}^* / \mu_k$. Hence $H^1(H, \mathbb{C}^* / \mu_m) = \{1\}$. By Lemma 4.7, we obtain $H^1(H, \mathbb{C}[x_0, \dots, x_n]_k) = \{1\}$ and thus $H^1(H, \mathrm{Aut}_{\mathbb{P}^n}(F_l)) = \{1\}$. So, all sections of π are conjugate.

Now, since H is simple and not contained in $\mathrm{Aut}_{\mathbb{P}^n}^0(F_l)$, we obtain $\pi \circ \phi(H) \subset H$. Both ϕ and π are algebraic morphisms, so $\pi \circ \phi(H) = H$. Therefore, up to the algebraic automorphism $\pi \circ \phi$, the homomorphism ϕ is a section of π . \square

4.3.2. Non conjugacy. It remains to show that the actions from Theorem 4.1 are not birationally conjugate.

Let M be a variety of dimension $n+1$ on which $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts faithfully.

If M is not rational, then M is isomorphic to $\mathbb{P}^n \times C$ for some smooth curve C . Recall that $\mathbb{P}^n \times C$ is birationally equivalent to $\mathbb{P}^n \times C'$ for smooth curves C and C' if and only if C and C' are birationally equivalent which again implies that C and C' are isomorphic. So, if $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts rationally and non trivially on a non rational variety M of dimension $n+1$, then this one is uniquely determined up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ and up to birational conjugation in $\mathrm{Bir}(M)$.

In the case that M is rational, we have to show that the $\mathrm{PGL}_{n+1}(\mathbb{C})$ -actions listed in Theorem 4.1 are not conjugate to each other. For this, note that none of them

has an orbit of codimension ≥ 1 . Lemma 3.2 induces therefore that any birational transformation conjugating one action to another one must be an isomorphism. As the varieties listed in Theorem 4.1 are not isomorphic we conclude that the actions are not conjugate.

5. EXTENSION TO Cr_n

In this section we study how the $\text{PGL}_{n+1}(\mathbb{C})$ -actions described in the above section extend to rational Cr_n -actions. Our goal is to prove Theorem 1.11. We proceed case by case.

5.1. **The case $\mathbb{G}(1, 3)$.** Let

$$s_1 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{ and } s_2 := \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \in \text{GL}_3(\mathbb{Z}).$$

Lemma 5.1. *Let G be a group. There exists no group homomorphism $\rho: \text{GL}_3(\mathbb{Z}) \rightarrow G$ such that $\rho(s_1)$ has order 3 and $s_2 \in \ker(\rho)$.*

Proof. Assume that such a ρ exists. Let

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, B := \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \text{GL}_3(\mathbb{Z}).$$

One calculates $(A(s_2(Bs_2B^{-1}))A^{-1}) = s_1T$. So s_1T is contained in the kernel of ρ and we get $\rho(T) = \rho(s_1^{-1})$. But this is a contradiction since the order of T is 2. \square

The following construction comes up in the context of tetrahedral line complexes (see [Dol12]). Consider the 4 hyperplanes in \mathbb{P}^3

$$E_0 := \{x_0 = 0\}, E_1 := \{x_1 = 0\}, E_2 := \{x_2 = 0\}, E_3 := \{x_3 = 0\}.$$

A line $l \in \mathbb{G}(1, 3)$ that is not contained in any of the E_i , intersects each plane E_i in one point p_i . We thus obtain a rational surjective map

$$cr: \mathbb{G}(1, 3) \dashrightarrow \mathbb{P}^1$$

that is defined by associating to the line l the cross ratio between the points p_i .

The closure $\overline{cr^{-1}([a : b])}$ in $\mathbb{G}(1, 3)$ is irreducible if and only if $[a : b] \in \mathbb{P}^1 \setminus \{[0 : 1], [1 : 0], [1 : 1]\}$, whereas $cr^{-1}([a : b])$ consists of two irreducible components in all the other cases ([Dol12, Chapter 10.3.6]).

Recall that α is the automorphism of PGL_4 given by $g \mapsto {}^t g^{-1}$.

Proposition 5.2. *There exists no non-trivial group homomorphism*

$$\Phi: \langle \text{PGL}_4(\mathbb{C}), W_3 \rangle \rightarrow \text{Bir}(\mathbb{G}(1, 3))$$

such that $\Phi(\text{PGL}_4(\mathbb{C})) \subset \text{Aut}(\mathbb{G}(1, 3))$.

In particular, neither the action of $\text{PGL}_4(\mathbb{C})$ on $\mathbb{G}(1, 3)$ given by the embedding φ_G (see Example 1.10) nor the action given by $\varphi_G \circ \alpha$ can be extended to a rational action of Cr_4 .

Proof. The proof of Corollary A.11 implies that if $\mathrm{PGL}_4(\mathbb{C})$ is contained in the kernel of a homomorphism $\Phi: \langle \mathrm{PGL}_4(\mathbb{C}), W_3 \rangle \rightarrow \mathrm{Bir}(\mathbb{G}(1, 3))$, then Φ is trivial. So we may assume that Φ embeds $\mathrm{PGL}_4(\mathbb{C})$ into $\mathrm{Aut}^0(\mathbb{G}(1, 3))$. By Theorem 4.1, it is therefore enough to show that φ_G and $\varphi_G \circ \alpha$ do not extend to a homomorphism of $\langle \mathrm{PGL}_4(\mathbb{C}), W_3 \rangle$.

The $\varphi_G(D_3)$ -orbit of a line that is not contained in one of the planes E_i and that does not pass through any of the points $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0], [0 : 0 : 0 : 1]$, has dimension 3 and these are all $\varphi_G(D_3)$ -orbits of dimension 3.

Since $\varphi_G(D_3)$ stabilizes the hyperplanes E_i and since the cross ratio is invariant under linear transformations, we obtain that cr is $\varphi_G(D_3)$ -invariant. By the above remark, the rational map cr therefore parametrizes all but finitely many $\varphi_G(D_3)$ -orbits of dimension 3 by $\mathbb{P}^1 \setminus \{[0 : 1], [1 : 0], [1 : 1]\}$.

The image $\varphi_G(\mathcal{S}_4)$, where $\mathcal{S}_4 \subset \mathrm{PGL}_4$ is the subgroup of coordinate permutations, normalizes $\varphi_G(D_3)$ and therefore it permutes its 3-dimensional orbits. Since \mathcal{S}_4 permutes the hyperplanes E_i , we can describe its action on the 3-dimensional $\varphi_G(D_3)$ -orbits by its action on the cross ratio of the intersection of general lines with the planes E_i .

Let r be the cross ratio between the points p_0, p_1, p_2, p_3 on a line. One calculates that the cross ratio between p_3, p_1, p_2, p_0 is again r and that the cross ratio between the points p_2, p_1, p_2, p_3 is $\frac{1}{1-r}$. Hence the image of $\tau_1 := [x_3 : x_1 : x_2 : x_0]$ leaves cr invariant, whereas for the permutation $\tau_2 := [x_2 : x_1 : x_0 : x_3]$ we have $cr \circ \varphi(\tau_2) \neq cr$ and $cr \circ \varphi(\tau_2)^2 \neq cr$.

Let $f: \mathbb{G}(1, 3) \dashrightarrow \mathbb{P}^4$ be a birational transformation and let $\varphi'_G := f \circ \varphi_G \circ f^{-1}$. The image $\varphi'_G(D_3) \subset \mathrm{Cr}_4$ is an algebraic torus of rank 3 and therefore, by Proposition 2.5, conjugate to the standard subtorus $D_3 \subset D_4$ of rank 3. In other words, there exists a rational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^1$ whose fibers consist of the closure of the $\varphi'_G(D_3)$ -orbits. The image $\varphi'_G(\mathcal{S}_4)$ permutes the torus orbits, hence we obtain a homomorphism $\rho: \mathcal{S}_4 \rightarrow \mathrm{PGL}_2(\mathbb{C})$. By what we observed above, the permutation τ_1 is contained in the kernel of ρ , whereas the image $\rho(\tau_2)$ has order 3. The matrix representation in $\mathrm{GL}_3(\mathbb{Z})$ of τ_1 corresponds to s_1 and the matrix representation of τ_2 corresponds to s_2 .

It follows now from Lemma 5.1 that ρ can not be extended to a homomorphism from $\mathrm{GL}_3(\mathbb{Z}) \simeq W_3$ to $\mathrm{PGL}_2(\mathbb{C})$, which implies that there exists no homomorphism $\Phi: \langle \mathrm{PGL}_4(\mathbb{C}), W_3 \rangle \rightarrow \mathrm{Cr}_4$ such that $\Phi(\mathrm{PGL}_4(\mathbb{C})) = \varphi'_G(\mathrm{PGL}_4(\mathbb{C}))$, since W_3 normalizes the torus and its image would therefore permute the torus orbits as well. The statement follows. \square

5.2. The case $\mathbb{P}(T\mathbb{P}^2)$. Recall that matrices of order two in $\mathrm{PGL}_2(\mathbb{C})$ have the form

$$(1) \quad \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix}, \text{ where } a \in \mathbb{C}^*, b, c \in \mathbb{C}, bc \neq -1.$$

Proposition 5.3. *The embedding $\varphi_B: \mathrm{PGL}_3(\mathbb{C}) \rightarrow \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2))$ extends uniquely to an embedding*

$$\Phi_B: \mathrm{Cr}_2 \rightarrow \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2)).$$

Proof. It is enough to show that every extension coincides with the one given in Example 1.9. For this it is enough to show that the image of σ is uniquely determined. Assume that there is an extension $\psi: \mathrm{Cr}_2 \rightarrow \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2))$ of φ_B . We will show $\psi = |\Psi_B$.

Let $d \in D_2$, $d = (ax_1, bx_2)$ with respect to affine coordinates given by $x_0 = 1$. Then $\varphi_B(d) = (ax_1, bx_2, (b/a)x_3)$, with respect to suitable local affine coordinates of $\mathbb{P}(T\mathbb{P}^2)$. Let $\phi: \mathbb{P}(T\mathbb{P}^2) \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ be the birational map given by

$$\phi: (x_1, x_2, x_3) \mapsto (x_1, x_2, \frac{x_1}{x_2}x_3),$$

with respect to local affine coordinates.

Let $\psi_1: \text{Cr}_2 \rightarrow \text{Bir}(\mathbb{P}^2 \times \mathbb{P}^1)$ be the algebraic embedding $\psi_1 = \phi \circ \psi \circ \phi^{-1}$. This gives us a \mathbb{P}^2 -fibration, which we call the horizontal fibration, and a \mathbb{P}^1 -fibration, which we call the vertical fibration. The image $\psi_1(D_2)$ acts canonically on the first factor and leaves the second one invariant. The horizontal fibers thus consist of the closures of D_2 -orbits. Since W_2 normalizes D_2 , the image $\psi_1(W_2)$ permutes the orbits of $\psi_1(D_2)$. Hence it preserves the horizontal fibration and we obtain a homomorphism

$$\rho: W_2 \simeq \text{GL}_2(\mathbb{Z}) \rightarrow \text{Bir}(\mathbb{P}^1) = \text{PGL}_2(\mathbb{C}).$$

In what follows we identify W_2 with $\text{GL}_2(\mathbb{Z})$.

The images of the three transpositions in $\mathcal{S}_3 = W_2 \cap \text{PGL}_3(\mathbb{C})$ under ρ are:

$$\rho \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \rho \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$\text{and } \rho \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The image $\rho(\sigma)$ is either the identity or it has order 2. The elements of the form (1) do not commute with the images of \mathcal{S}_3 described above. Since σ is contained in the center of W_2 , we obtain $\rho(\sigma) = \text{id}$.

It remains to show that the action of $\psi_1(\sigma)$ on the first factor of $\mathbb{P}^2 \times \mathbb{P}^1$ is the standard action. Let $M = \mathbb{P}^2$ be a horizontal fiber. It is stabilized by $\psi(D_2)$ and $\psi(\sigma)$, so we obtain a homomorphism

$$\gamma: \langle D_2, \sigma \rangle \rightarrow \text{Bir}(M) = \text{Cr}_2.$$

Since $\sigma d \sigma^{-1} = d^{-1}$ for all $d \in D_2$, there exists a $d \in D_2$ such that $\gamma(\sigma) = d\sigma$. This is true for all horizontal fibers, so $\psi_1(\sigma)$ induces an automorphism of $U \times \mathbb{P}^1$, where

$$U = \{[x_0 : x_1 : x_2] \mid x_0, x_1, x_2 \neq 0\} \subset \mathbb{P}^2.$$

Let $S \simeq \mathbb{P}^1 \subset U \times \mathbb{P}^1$ be a vertical fiber and $\pi: U \times \mathbb{P}^1 \rightarrow U$ the projection onto the first factor. Then $\pi \circ \psi_1(\sigma)(S)$ is a regular map from \mathbb{P}^1 to the affine set U and is therefore constant. We obtain that $\psi_1(\sigma)$ preserves the vertical fibration.

The image $\psi_1(\text{PGL}_3(\mathbb{C}))$ preserves the vertical fibration as well and projection onto \mathbb{P}^2 yields a homomorphism from $\text{PGL}_3(\mathbb{C})$ to Cr_2 that is the standard embedding. Hence $\psi_1(\text{Cr}_2)$ preserves the vertical fibration and we obtain an algebraic homomorphism from Cr_2 to Cr_2 , which is uniquely determined by its restriction to $\text{PGL}_3(\mathbb{C})$. So the image $\psi_1(\sigma)$ is uniquely determined. \square

Lemma 5.4. *There exists no homomorphism $\Phi: \text{Cr}_2 \rightarrow \text{Cr}_3$ such that*

$$\Phi|_{\text{PGL}_3(\mathbb{C})} = \varphi_B \circ \alpha,$$

where φ_B denotes the embedding of PGL_3 into Cr_3 from Example 1.9 and α the algebraic automorphism of PGL_3 given by $g \mapsto {}^t g^{-1}$.

Proof. Assume that such an extension $\Phi: \text{Cr}_2 \rightarrow \text{Cr}_3$ of $\varphi_B \circ \alpha$ exists.

Observe that $\alpha(D_2) = D_2$ and that $\alpha|_{S_3} = \text{id}_{S_3}$. Therefore, we can repeat the same argument as in the proof of Proposition 5.3 to obtain $\Psi(\sigma) = \Phi_B(\sigma)$. But we have

$$\Psi(\sigma)\Psi(g)\Psi(\sigma)\Psi(g)\Psi(\sigma)\Psi(g) \neq \text{id}$$

for $g = [z - x : z - y : z]$ - this contradicts the relations in Cr_2 (Proposition A.9). \square

5.3. The case $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k(n+1)))$.

Proposition 5.5. *The algebraic homomorphism $\varphi_l: \text{PGL}_{n+1}(\mathbb{C}) \rightarrow \text{Bir}(F_l)$ extends uniquely to the embedding*

$$\Psi_l: \langle \text{PGL}_{n+1}(\mathbb{C}), W_n \rangle \rightarrow \text{Bir}(F_l) \text{ (see Example 1.8).}$$

Proof. Suppose that there is an extension $\psi: H_n \rightarrow \text{Bir}(F_l)$ of φ_l . We will show that ψ is unique and therefore that $\psi = \Psi_l$.

Let (x_1, \dots, x_{n-1}, w) be local affine coordinates of F_l such that for every $g \in \text{PGL}_{n+1}(\mathbb{C})$ the image $\varphi_l(g)$ acts by

$$(x_1, \dots, x_n, w) \mapsto (g(x_1, \dots, x_n), J(g(x_1, \dots, x_n))^{-l}w).$$

In particular, the image under ψ of $(d_1x_1, \dots, d_nx_n) \in D_n$ acts by

$$(x_1, \dots, x_n, w) \mapsto (d_1x_1, \dots, d_nx_n, (d_1 \cdots d_n)^{-l}w).$$

Define $\phi: F_l \rightarrow \mathbb{P}^n \times \mathbb{P}^1$ by

$$\phi: (x_1, \dots, x_n, w) \mapsto (x_1, \dots, x_n, (x_1 \cdots x_n)^l w)$$

with respect to local affine coordinates. Let $\psi_1: \text{Cr}_n \rightarrow \text{Bir}(\mathbb{P}^n \times \mathbb{P}^1)$ be the algebraic embedding $\psi_1 := \phi \circ \psi \circ \phi^{-1}$. Then the image $\psi_1(D_n)$ acts canonically on the first factor and leaves the second one invariant. Since W_n normalises D_n , the image $\psi_1(W_n)$ permutes the orbits of $\psi_1(D_n)$. Hence $\psi_1(W_n)$ preserves the horizontal fibration. This induces a homomorphism

$$\rho: W_n \simeq \text{GL}_n(\mathbb{Z}) \rightarrow \text{PGL}_2(\mathbb{C}).$$

In what follows, we identify W_n with $\text{GL}_n(\mathbb{Z})$. Let $\mathcal{A}_{n+1} \subset \mathcal{S}_{n+1} \subset \text{PGL}_{n+1}(\mathbb{C})$ be the subgroup of coordinate permutations $s \in \mathcal{S}_{n+1}$ such that $J(s) = 1$. Hence $\mathcal{A}_{n+1} \in \ker(\rho)$. Note that the fixed point set of $\psi_1(\mathcal{A}_{n+1})$ is the vertical fiber

$$L := [1 : \cdots : 1] \times \mathbb{P}^1 \subset \mathbb{P}^n \times \mathbb{P}^1.$$

Since σ_n commutes with \mathcal{A}_{n+1} , the image $\psi_1(\sigma_n)$ stabilises L . The group $\psi_1(D_n)$ acts transitively on an open dense subset of vertical fibers that contains L . Since $\psi_1(\sigma_n)$ normalizes $\psi_1(D_n)$, we obtain that $\psi_1(\sigma_n)$ preserves the vertical fibration. Therefore $\langle \text{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$ preserves the vertical fibration. We obtain a homomorphism $\langle \text{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle \rightarrow \text{Cr}_n$, which is, by Corollary A.12 and its proof, the standard embedding.

Let

$$f_A = \left(\frac{1}{x_1}, x_2, \dots, x_n \right).$$

In [BH14] it is shown that f_A is contained in $\langle \text{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$, which implies that $\psi_1(f_A)$ preserves the vertical fibration and that its action on \mathbb{P}^n is the standard action.

Recall that $(hf_A)^3 = \text{id}$ for $h = (1 - x_1, x_2, \dots, x_{n-1}) \in \text{Cr}_n$. The image $\psi_1(h)$ is

$$\psi_1(h): (x_1, \dots, x_n, z) \mapsto (1 - x_1, x_2, \dots, x_n, (-1)^l z).$$

Denote by $A \in \text{GL}_n(\mathbb{Z})$ the integer matrix corresponding to f_A . We have $\rho(A) = \text{id}$ or $\rho(A)$ is of order two, i.e. has the form (1).

Suppose that $\rho(A) = \text{id}$. Then

$$\psi_1(f_A): (x_1, \dots, x_n, z) \mapsto \left(\frac{1}{x_1}, x_2 \dots x_n, z\right).$$

The relation $(hf_A)^3 = \text{id}$ then implies that l is even.

Suppose that

$$\rho(f_A) = \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix}, \text{ where } b, c \in \mathbb{C}, bc \neq -1,$$

hence

$$\psi_1(f_A): (x_1, \dots, x_n, z) \mapsto \left(\frac{1}{x_1}, x_2 \dots x_n, \frac{z+b}{cz-1}\right)$$

and therefore

$$\psi_1(hf_A) = (x_1, \dots, x_n, z) \mapsto \left(1 - \frac{1}{x_1}, \dots, x_n, \frac{(-1)^l z + (-1)^l b}{cz-1}\right),$$

One calculates that if l is even, then the relation $(hf_A)^3 = \text{id}$ is not satisfied. So assume that l is odd. This gives

$$\psi_1(hf_A)^3 = (x_1, \dots, x_n, z) \mapsto \left(x_1, \dots, x_n, \frac{a_1 z + a_2}{a_3 z - a_4}\right),$$

where $a_1 = 3bc - 1$, $a_2 = (bc - 1)b - 2b$, $a_3 = (1 - bc)c + 2c$ and $a_4 = 3bc - 1$. So $(hf_A)^3 = \text{id}$ yields either l odd and $b = c = 0$ or l odd and $bc = 3$. However, the latter is not possible. Consider the transformation

$$\tau = (x_1, \dots, x_{n-2}, x_n, x_{n-1}) \in \mathcal{S}_n.$$

We have $f_A \tau = \tau f_A$. Note that

$$\psi_1(\tau): (x_1, \dots, x_n, z) \mapsto (x_1, \dots, x_{n-2}, x_n, x_{n-1}, \dots, x_n, (-1)^l z)$$

and this transformation does not commute with $\left(x_1, \dots, x_n, \frac{a_1 z + a_2}{a_3 z - a_4}\right)$ in the second case. Hence $c = b = 0$ and l is odd.

Finally, assume that

$$\rho(f_A) = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \text{ where } a \in \mathbb{C}^*.$$

This implies

$$\psi_1(f_A): (x_1, \dots, x_n, z) \mapsto \left(\frac{1}{x_1}, x_2 \dots x_n, \frac{1}{az}\right)$$

and hence $\psi(hf_A)^3 \neq \text{id}$.

We conclude that

$$\rho(f_A) = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^l \end{bmatrix}$$

and therefore that the action of $\psi(f_A)$ is uniquely determined by l . Hence

$$\psi|_{\langle \text{PGL}_n(\mathbb{C}), \sigma_{n-1} \rangle} = \Psi_l|_{\langle \text{PGL}_n(\mathbb{C}), \sigma_{n-1} \rangle}.$$

Let f_B, f_C, f_D and $f_E \in \text{Cr}_n$ be as in the proof of Corollary A.11. By Lemma A.10 it remains to show that the image $\psi(f_B)$ is uniquely determined. We use once more the relation

$$f_B = f_D f_C f_E f_D^{-1}.$$

Since $\rho(CE) = \text{id}$ and since f_D has order two, we obtain $\rho(B) = \text{id}$.

Let $c \in \mathbb{P}^1$ such that the restriction of $\psi_1(f_B)$ to the hyperplane

$$\{c\} \times \mathbb{P}^n \subset \mathbb{P}^1 \times \mathbb{P}^n$$

is a birational map. Then the restriction of $\psi_1(f_B)$ to $\{c\} \times \mathbb{P}^n$ has to fulfill the relations with the group $\langle \text{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. By Corollary A.12 we obtain that this restriction has to be f_B . Hence the image $\psi_1(f_B)$ is unique. \square

Proposition 5.6. *There exists no group homomorphism $\psi: \text{H}_n \rightarrow \text{Bir}(F_l)$ such that $\psi|_{\text{PGL}_{n+1}(\mathbb{C})} = \varphi_l \circ \alpha$.*

Proof. Assume that such an extension $\psi: \text{H}_n \rightarrow \text{Cr}_n$ exists. Let ϕ be as in Proposition 5.5 and $\psi_2: \text{H}_n \rightarrow \text{Bir}(\mathbb{P}^1 \times \mathbb{P}^n)$,

$$\psi_2 := \phi \circ \varphi_l \circ \alpha \circ \phi^{-1}.$$

Similarly as in the proof of Proposition 5.5 one can show that $\psi_2(\sigma_n)$ preserves the vertical fibration. In that way we obtain an algebraic homomorphism

$$A: \langle \text{PGL}_{n+1}, \sigma \rangle \rightarrow \text{Cr}_n$$

such that $A|_{\text{PGL}_{n+1}(\mathbb{C})} = \alpha$. Such a homomorphism does not exist by Corollary A.12. \square

5.4. The case $C \times \mathbb{P}^n$.

Proposition 5.7. *The embedding $\varphi_C: \text{PGL}_{n+1}(\mathbb{C}) \rightarrow \text{Bir}(C \times \mathbb{P}^n)$ extends uniquely to the standard embedding*

$$\Phi_C: \text{H}_n \rightarrow \text{Bir}(C \times \mathbb{P}^n) \text{ (see Example 5.7).}$$

Proof. Suppose that there is an extension $\Psi: \text{H}_n \rightarrow \text{Bir}(C \times \mathbb{P}^n)$ of φ_C . By definition, $\Psi(\text{PGL}_{n+1}(\mathbb{C}))$ fixes the horizontal fibration with fibers isomorphic to \mathbb{P}^n . Moreover, each of the horizontal fibers is a closure of a $\Psi(D_n)$ -orbit. Since the elements of W_n commute with D_n , we conclude that $\Psi(W_n)$ preserves the horizontal fibration. Hence H_n preserves the horizontal fibration and we obtain a homomorphism

$$\rho: \text{H}_n \rightarrow \text{Bir}(C)$$

such that $\text{PGL}_{n+1}(\mathbb{C}) \subset \ker(\rho)$. In the Appendix it is shown that the normal subgroup generated by PGL_{n+1} in H_n is all of H_n . Hence ρ is trivial and $\Psi(\text{H}_n)$ fixes the horizontal fibration. The restriction $\Psi(\text{H}_n)|_{C \times \mathbb{P}^n}$ for any $c \in C$ defines a homomorphism from H_n to Cr_n such that the restriction to $\text{PGL}_{n+1}(\mathbb{C})$ is the standard embedding. By Corollary A.12, this is the standard embedding. Hence Ψ is unique. \square

Proposition 5.8. *There exists no group homomorphism $\Psi: \text{H}_n \rightarrow \text{Bir}(\mathbb{C} \times \mathbb{P}^n)$ such that $\Psi|_{\text{PGL}_{n+1}(\mathbb{C})} = \varphi_C \circ \alpha$.*

Proof. Assume there exists such a Ψ . As in the proof of Proposition 5.7 one can show that $\Psi(H_n)$ fixes the horizontal fibration. The restriction $\Psi(H_n)|_{C \times \mathbb{P}^{n-1}}$ defines for each $c \in C$ a homomorphism from H_n to Cr_n such that the restriction to $\text{PGL}_{n+1}(\mathbb{C})$ is given by $g \mapsto \alpha(g)$. By Corollary A.12, there exists no such homomorphism. \square

APPENDIX

Relations and structures in Cr_n . We will often use the following relations between elements of the Cremona group:

Lemma A.9. *In Cr_2 the following relations hold:*

- (1) $\sigma\tau(\tau\sigma)^{-1} = \text{id}$ for all $\tau \in \mathcal{S}_3$,
- (2) $\sigma d = d^{-1}\sigma$ for all diagonal maps $d \in D_2$ and
- (3) $(\sigma h)^3 = \text{id}$ for $h = [x_2 - x_0 : x_2 - x_1 : x_2]$.

Proof. One checks the identities by direct calculation. \square

Denote by $\text{Cr}_n^0 \subset \text{Cr}_n$ the subgroup consisting of elements that contract only rational hypersurfaces. We have $H_n \subset \text{Cr}_n^0$. On the other hand, it seems to be an interesting open question, whether there exist elements in Cr_n^0 that are not contained in H_n for any $n \geq 3$ (cf. [Lam14]).

Lemma A.10. *The group H_n is generated by $\text{PGL}_{n+1}(\mathbb{C})$ and the birational transformations $\sigma_n := (x_1^{-1}, x_2^{-1}, \dots, x_n^{-1})$ and $f_B := (x_1 x_2, x_3, \dots, x_n)$.*

Proof. It is known that $\text{GL}_n(\mathbb{Z})$ is generated by the subgroup of permutation matrices in $\text{GL}_n(\mathbb{Z})$ and the two elements

$$A := \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

(see for example [dlH00, III.A.2]). Notice that f_B is the birational transformation in W_n corresponding to B . Let f_A be the birational transformation corresponding to A . In [BH14] it is shown that f_A is contained in $\langle \text{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. \square

The goal of this appendix is to prove the following two corollaries of Theorem 1.3:

Corollary A.11. *Let $n > m$ and let $\Phi: \text{Cr}_n \rightarrow \text{Cr}_m$ be a group homomorphism. Then the normal subgroup of Cr_n containing H_n is contained in the kernel of Φ .*

No such non-trivial homomorphism is known so far. In fact, it is an open question, whether Cr_n is simple for $n \geq 3$.

Corollary A.12. *Let $\Psi: H_n \rightarrow \text{Cr}_n$ be a non-trivial group homomorphism. Then there exists an automorphism of fields γ of \mathbb{C} and an element $g \in \text{Cr}_n$ such that $\gamma(g\Psi g^{-1})$ is the standard embedding.*

Moreover, the extension of the standard embedding $\varphi: \text{PGL}_{n+1} \rightarrow \text{Cr}_n$ to the group H_n is unique. The embedding $\varphi \circ \alpha$, where $\alpha: \text{PGL}_{n+1} \rightarrow \text{PGL}_{n+1}$ is the algebraic automorphism $g \mapsto {}^t g^{-1}$, does not extend to a homomorphism from H_n to Cr_n .

By the theorem of Noether and Castelnuovo, Corollary A.12 implies in particular the theorem of Déserti about automorphisms of Cr_2 .

Proof of Corollary A.11. By Lemma A.10 it is enough to show that σ_n and f_B are contained in the normal subgroup containing $\text{PGL}_{n+1}(\mathbb{C})$. Let

$$g_n := [x_n - x_0 : x_n - x_1 : \cdots : x_n - x_{n-1} : x_n] \in \text{PGL}_{n+1}(\mathbb{C}).$$

Then $\sigma_n g_n \sigma_n g_n \sigma_n g_n = \text{id}$. In particular, $\sigma_n g_n$ conjugates σ_n to g_n .

Let

$$C := \begin{pmatrix} -1 & 2 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, D := \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$E := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \text{GL}_n(\mathbb{Z})$$

and let f_C, f_D and f_E be the corresponding elements in W_n . It is shown in [BH14] that f_C is contained in $\langle \text{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. Moreover, one calculates that

$$f_B = f_D f_C f_E f_D^{-1},$$

which implies that f_B is conjugate to an element in $\langle \text{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. \square

Proof of Corollary A.12. By Theorem 1.3 we may assume that, up to conjugation and automorphism of fields, the restriction of Ψ to $\text{PGL}_{n+1}(\mathbb{C})$ is the standard embedding or the standard embedding composed with the automorphism α of $\text{PGL}_{n+1}(\mathbb{C})$ given by $\alpha(g) = {}^t g^{-1}$.

In particular, $\Psi(D_n) = D_n$ and therefore $\Psi(W_n)$ is contained in $D_n \rtimes W_n$. Assume that $\Psi(\sigma_n) = d\tau$ for some $d \in D_n$ and $\tau \in W_n$. The relation $\Psi(\sigma_n)e\Psi(\sigma_n) = e^{-1}$ for all $e \in D_n$ implies $\tau = \sigma_n$. Note that the restriction of Ψ to \mathcal{S}_{n+1} is the standard embedding. So for all $\tau \in \mathcal{S}_{n+1}$ we obtain

$$\tau d \sigma_n = d \sigma_n \tau = d \tau \sigma_n.$$

The only element in D_n that commutes with \mathcal{S}_{n+1} is the identity. Hence $\Psi(\sigma_n) = \sigma_n$.

Let g_n be as in the proof of Corollary A.11. The relation $\sigma_n g_n \sigma_n g_n \sigma_n g_n = \text{id}$ implies that $\Psi|_{\text{PGL}_{n+1}(\mathbb{C})}$ is the standard embedding, since

$$\sigma_n \alpha(g_n) \sigma_n \alpha(g_n) \sigma_n \alpha(g_n) \neq \text{id}.$$

It remains to show that $\Psi(f_B) = f_B$. Let $d \in D_n$, and $\rho \in W_n$ such that $\Psi(f_B) = d\rho$. The image $\Psi(f_B)$ acts on $\Psi(D_n)$ by conjugation. We have that $\Psi(D_n) = D_n$, so the action of $\Psi(f_B)$ on $\Psi(D_n)$ is determined by ρ . Since $\Psi|_{D_n}$ is the standard embedding, we obtain $\rho = f_B$. Let $d = (d_1 x_1, \dots, d_n x_n)$. The image $\Psi(f_B)$ commutes with σ_n . We obtain

$$d^{-1} \sigma_n f_B = \sigma_n d f_B = d f_B \sigma_n = d \sigma_n f_B$$

and hence $d_i = \pm 1$ for all i .

The image $\Psi(f_B)$ commutes with all elements of \mathcal{S}_{n+1} that fix the coordinates x_1 and x_2 . Similar as above, this yields that d commutes with all elements of \mathcal{S}_{n+1} that fix the coordinates x_1 and x_2 and we get $d_i = 1$ for $i \neq 1$ and $i \neq 2$.

In [BH14] it is shown that f_B^2 is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. By what we proved above, this gives

$$\Psi(f_B^2) = f_B^2 = df_B df_B = dd' f_B^2,$$

where $d' = (d_1 d_2 x_1, d_2 x_2, \dots, d_n x_n)$. So $dd' = \mathrm{id}$, which yields $d_1^2 d_2 = 1$ and therefore $d_2 = 1$. This means that we have either $\Psi(f_B) = f_B$ or $\Psi(f_B) = df_B$ with $d = (-x_1, x_2, \dots, x_n)$.

Let

$$\begin{aligned} r_1 &:= [x_0 : x_1 : \dots : x_{n-1} : x_n + x_1], r_2 := [x_n : x_1 : \dots : x_{n-1} : x_0], \\ r_3 &:= [x_n : x_0 : x_2 : \dots : x_{n-1} : x_1], t := [x_n : x_0 : \dots : x_{n-1}]. \end{aligned}$$

We have the relation

$$(r_2 t f_B t^{-1} r_3) r_1 (r_2 t f_B t^{-1} r_3) = r_1$$

and therefore

$$(r_2 t \Psi(f_B) t^{-1} r_3) r_1 (r_2 t \Psi(f_B) t^{-1} r_3) = r_1.$$

One calculates that if $\Psi(f_B) = (-x_1, x_2, \dots, x_n) f_B$ then this relation is not satisfied. Hence $\Psi(f_B) = f_B$. \square

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